Zeros of a Polynomial Function

An important consequence of the Factor Theorem is that finding the zeros of a polynomial is really the same thing as factoring it into linear factors. In this section we will study more methods that help us find the real zeros of a polynomial, and thereby factor the polynomial.

Rational Zeros of Polynomials:

The next theorem gives a method to determine all possible candidates for rational zeros of a polynomial function with integer coefficients.

Rational Zeros Theorem:

If the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients, then every rational zero of P is of the form

$$\frac{p}{q}$$

where p is a factor of the constant coefficient a_0

and q is a factor of the leading coefficient a_n

Example 1: List all possible rational zeros given by the Rational Zeros Theorem of $P(x) = 6x^4 + 7x^3 - 4$ (but don't check to see which actually are zeros) .

Solution:

Step 1: First we find all possible values of p, which are all the factors of $a_0 = 4$. Thus, p can be ±1, ±2, or ±4.

Step 2: Next we find all possible values of q, which are all the factors of $a_n = 6$. Thus, q can be ±1, ±2, ±3, or ±6.

Step 3: Now we find the possible values of $\frac{p}{q}$ by making combinations of the values we found in Step 1 and Step 2. Thus, $\frac{p}{q}$ will be of the form $\frac{\text{factors of } 4}{\text{factors of } 6}$. The possible $\frac{p}{q}$ are

$$\pm\frac{1}{1},\ \pm\frac{2}{1},\ \pm\frac{4}{1},\ \pm\frac{1}{2},\ \pm\frac{2}{2},\ \pm\frac{4}{2},\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{1}{6},\ \pm\frac{2}{6},\pm\frac{4}{6}$$

Example 1 (Continued):

> **Step 4:** Finally, by simplifying the fractions and eliminating duplicates, we get the following list of possible values for $\frac{p}{q}$.
>
> $$\pm 1, \ \pm 2, \ \pm 4, \ \pm \frac{1}{2}, \ \pm \frac{1}{3}, \ \pm \frac{2}{3}, \ \pm \frac{4}{3}, \ \pm \frac{1}{6}$$

Now that we know how to find all possible rational zeros of a polynomial, we want to determine which candidates are actually zeros, and then factor the polynomial. To do this we will follow the steps listed below.

> **Finding the Rational Zeros of a Polynomial:**
>
> 1. **Possible Zeros:** List all possible rational zeros using the Rational Zeros Theorem.
>
> 2. **Divide:** Use Synthetic division to evaluate the polynomial at each of the candidates for rational zeros that you found in Step 1. When the remainder is 0, note the quotient you have obtained.
>
> 3. **Repeat:** Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

Example 2: Find all real zeros of the polynomial $P(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$.

> **Solution:**
>
> > **Step 1:** First list all possible rational zeros using the Rational Zeros Theorem. For the rational number $\frac{p}{q}$ to be a zero, p must be a factor of $a_0 = 2$ and q must be a factor of $a_n = 2$. Thus the possible rational zeros, $\frac{p}{q}$, are
> >
> > $$\pm 1, \ \pm 2, \ \pm \frac{1}{2}$$

Example 2 (Continued):

Step 2: Now we will use synthetic division to evaluate the polynomial at each of the candidates for rational zeros we found in Step 1. When we get a remainder of zero, we have found a zero.

$$
\begin{array}{r|rrrr}
1 & 2 & 1 & -6 & -7 & -2 \\
  &   & 2 & 3 & -3 & -10 \\
\hline
  & 2 & 3 & -3 & -10 & -12
\end{array}
$$
$\leftarrow$ Since the remainder is not zero,
$+1$ is not a zero

$$
\begin{array}{r|rrrr}
-1 & 2 & 1 & -6 & -7 & -2 \\
   &   & -2 & 1 & 5 & 2 \\
\hline
   & 2 & -1 & -5 & -2 & 0
\end{array}
$$
$\leftarrow$ Since the remainder is zero,
-1 is a zero

This also tells us that P factors as

$$2x^4 + x^3 - 6x^2 - 7x - 2 = (x + 1)(2x^3 - x^2 - 5x - 2)$$

Step 3: We now repeat the process on the quotient polynomial $2x^3 - x^2 - 5x - 2$. Again using the Rational Zeros Theorem, the possible rational zeros of this polynomial are

$$\pm 1, \ \pm 2, \ \pm \frac{1}{2}.$$

Since we determined that $+1$ was not a rational zero in Step 2, we do not need to test it again, but we should test -1 again.

$$
\begin{array}{r|rrr}
-1 & 2 & -1 & -5 & -2 \\
   &   & -2 & 3 & 2 \\
\hline
   & 2 & -3 & -2 & 0
\end{array}
$$
$\leftarrow$ Since the remainder is zero,
-1 is again a zero

Thus, P factors as

$$
\begin{aligned}
2x^4 + x^3 - 6x^2 - 7x - 2 &= (x + 1)(2x^3 - x^2 - 5x - 2) \\
&= (x + 1)(x + 1)(2x^2 - 3x - 2) \\
&= (x + 1)^2(2x^2 - 3x - 2)
\end{aligned}
$$

Example 2 (Continued):

Step 4: At this point the quotient polynomial, $2x^2 - 3x - 2$, is quadratic. This factors easily into $(x - 2)(2x + 1)$, which tells us we have zeros at $x = 2$ and $x = -\dfrac{1}{2}$, and that P factors as

$$
\begin{aligned}
2x^4 + x^3 - 6x^2 - 7x - 2 &= (x + 1)(2x^3 - x^2 - 5x - 2)\\
&= (x + 1)\,(x + 1)(2x^2 - 3x - 2)\\
&= (x + 1)^2\,(2x^2 - 3x - 2)\\
&= (x + 1)^2\,(x - 2)(2x + 1)
\end{aligned}
$$

Step 5: Thus the zeros of $P(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$ are $x = -1$, $x = 2$, and $x = -\dfrac{1}{2}$.

Descartes' Rule of Signs and Upper and Lower Bounds for Roots:

In many cases, we will have a lengthy list of possible rational zeros of a polynomial. A theorem that is helpful in eliminating candidates is Descartes' Rule of Signs.

In the theorem, **variation in sign** is a change from positive to negative, or negative to positive in successive terms of the polynomial. Missing terms (those with 0 coefficients) are counted as no change in sign and can be ignored. For example,

$$
P(x) = 3x^3 + x^2 - 15x + 2
$$

has two variations in sign.

Descartes' Rule of Signs: Let P be a polynomial with real coefficients

1. The number of positive real zeros of $P(x)$ is either equal to the number of variations in sign in $P(x)$ or is less than that by an even whole number.

2. The number of negative real zeros of $P(x)$ is either equal to the number of variations in sign in $P(-x)$ or is less than that by an even whole number.

Example 3: Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros $P(x) = 6x^3 + 17x^2 - 31x - 12$ can have. Then determine the possible total number of real zeros.

       Solution:

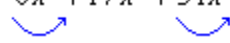   **Step 1:** First we will count the number of variations in sign of

$$P(x) = 6x^3 + 17x^2 - 31x - 12.$$

$$P(x) = 6x^3 + 17x^2 \underbrace{- 31x} - 12$$

Since there is only one variation, $P(x)$ has one positive real zero.

          Step 2: Now we will count the number of variations in sign of

$$P(-x) = -6x^3 + 17x^2 + 31x - 12.$$

$$P(-x) = \underbrace{-6x^3 + 17x^2} + \underbrace{31x - 12}$$

$P(-x)$ has two variations in sign, thus $P(x)$ has two or zero negative real zeros.

          Step 3: Finally by combining our findings in the previous steps, $P(x)$ has either one or three real zeros.

Positive	Negative	Real
1	0	1
1	2	3

Another theorem useful in eliminating candidates for real zeros of a polynomial is the Upper and Lower Bounds Theorem.

We say that a is a **lower bound** and b is an **upper bound** for the roots of a polynomial equation if every real root c of the equation satisfies $a \leq c \leq b$.

The Upper and Lower Bounds Theorem: Let P be a polynomial with real coefficients.

1. If we divide $P(x)$ by $x - b$ (with $b > 0$) using synthetic division, and if the row that contains the quotient and the remainder has no negative entry, then b is an upper bound for the real zeros of P.

2. If we divide $P(x)$ by $x - a$ (with $a < 0$) using synthetic division, and if the row that contains the quotient and the remainder has entries that are alternately nonpositive (negative or zero) and nonnegative (positive or zero), then a is a lower bound for the real zeros of P.

Example 4: Show that $a = -4$ and $b = 5$ are lower and upper bounds for the real zeros of the polynomial $P(x) = x^4 - 2x^3 - 14x^2 + 14x + 24$.

Solution:

Step 1: We will start to show $a = -4$ is a lower bound of $P(x) = x^4 - 2x^3 - 14x^2 + 14x + 24$, by dividing $P(x)$ by $x - (-4)$ using synthetic division.

$$
\begin{array}{r|rrrrr}
-4 & 1 & -2 & -14 & 14 & 24 \\
   &   & -4 & 24 & -40 & 104 \\
\hline
   & 1 & -6 & 10 & -26 & 128 \\
\end{array}
$$
$\leftarrow$ Entries alternate in sign

Step 2: By the Upper and Lower Bounds Theorem, -4 is a lower bound of $P(x)$, because the row that contains the quotient and remainder has entries that are alternately nonpositive and nonnegative.

Step 3: Now we will show $b = 5$ is an upper bound of $P(x) = x^4 - 2x^3 - 14x^2 + 14x + 24$. Divide $P(x)$ by $x - 5$ using synthetic division.

$$
\begin{array}{r|rrrrr}
5 & 1 & -2 & -14 & 14 & 24 \\
  &   & 5 & 15 & 5 & 95 \\
\hline
  & 1 & 3 & 1 & 19 & 119 \\
\end{array}
$$
$\leftarrow$ All entries positive

Step 4: By the Upper and Lower Bounds Theorem, 5 is an upper bound of $P(x)$, because the row that contains the quotient and remainder has no negative entry.

Example 5: Find all rational zeros of $P(x) = 6x^4 - 23x^3 + 3x^2 + 32x + 12$, and then find the irrational zeros, if any and graph the polynomial. If appropriate, use the Rational Zeros Theorem, the Upper and Lower Bounds Theorem, Descartes' Rule of Signs, the quadratic formula, or other factoring techniques.

Solution:

Step 1: First we will use Descartes' Rule of Signs to determine how many positive and how many negative real zeros $P(x) = 6x^4 - 23x^3 + 3x^2 + 32x + 12$ can have.

$$P(x) = 6x^4 - 23x^3 + 3x^2 + 32x + 12$$

$$P(-x) = 6x^4 + 23x^3 + 3x^2 - 32x + 12$$

Counting the number of variations in sign of $P(x)$ and $P(-x)$, we see that $P(x) = 6x^4 - 23x^3 + 3x^2 + 32x + 12$ has zero or two positive real zeros and zero or two negative real zeros, making a total of either zero, two or four real zeros.

Positive	Negative	Real
0	0	0
2	0	2
0	2	2
2	2	4

Step 2: Next, using the Rational Zeros Theorem, we list the possible rational zeros of P, and then begin testing candidates for zeros.

The possible rational zeros of P are

$$\pm\frac{1}{6}, \ \pm\frac{1}{3}, \ \pm\frac{1}{2}, \ \pm\frac{2}{3}, \ \pm 1, \ \pm\frac{4}{3}, \ \pm\frac{3}{2}, \ \pm 2, \ \pm 3, \ \pm 4, \ \pm 6, \ \pm 12$$

We will check the positive candidates first, beginning with the smallest.

Example 5 (Continued):

$$\frac{1}{6}\Big|\ 6\quad -23\quad\ \ 3\quad\ \ 32\quad\ \ 12$$

$$1\quad -\frac{11}{3}\quad -\frac{1}{9}\quad \frac{287}{54}$$

$$6\quad -22\quad -\frac{2}{3}\quad \frac{287}{9}\quad \frac{935}{54}\quad\leftarrow\ \frac{1}{6}\ \text{is not a zero}$$

$$\frac{1}{3}\Big|\ 6\quad -23\quad\ \ 3\quad\ \ 32\quad\ \ 12$$

$$2\quad -7\quad -\frac{4}{3}\quad \frac{92}{9}$$

$$6\quad -21\quad -4\quad \frac{92}{3}\quad \frac{200}{9}\quad\leftarrow\ \frac{1}{3}\ \text{is not a zero}$$

$$\frac{1}{2}\Big|\ 6\quad -23\quad\ \ 3\quad\ \ 32\quad\ \ 12$$

$$3\quad -10\quad -\frac{7}{2}\quad \frac{57}{4}$$

$$6\quad -20\quad -7\quad \frac{57}{2}\quad \frac{105}{4}\quad\leftarrow\ \frac{1}{2}\ \text{is not a zero}$$

… (we continue testing candidates until) …

$$2\Big|\ 6\quad -23\quad\ \ 3\quad\ \ 32\quad\ \ 12$$

$$12\quad -22\quad -38\quad -12$$

$$6\quad -11\quad -19\quad -6\quad\ \ 0\quad\leftarrow\ P(2)=0$$

So 2 is a zero, and $P(x) = (x-2)(6x^3 - 11x^2 - 19x - 6)$.

Step 3: We continue by factoring the quotient. Again use the Rational Zeros Theorem to list the possible rational zeros, and test candidates for zeros.

The possible rational zeros of the quotient are

$$\pm\frac{1}{6},\ \pm\frac{1}{3},\ \pm\frac{1}{2},\ \pm\frac{2}{3},\ \pm1,\ \pm\frac{3}{2},\ \pm2,\ \pm3,\ \pm6.$$

From Step 1 we know we cannot have just one positive real zero, so we will continue testing positive candidates. Also, in Step 2, we already tested $\frac{1}{6},\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ \frac{4}{3}$, and $\frac{3}{2}$ and found they were not zeros, so we do not need to test them again, but we do need to test 2 again.

Example 5 (Continued):

$$
\begin{array}{r|rrrr}
2 & 6 & -11 & -19 & -6 \\
  &   & 12 & 2 & -34 \\
\hline
  & 6 & 1 & -17 & -40
\end{array}
\leftarrow \text{2 is not a zero}
\qquad
\begin{array}{r|rrrr}
3 & 6 & -11 & -19 & -6 \\
  &   & 18 & 21 & 6 \\
\hline
  & 6 & 7 & 2 & 0
\end{array}
\leftarrow P(3)=0
$$

Thus 3 is a zero, and $P(x) = (x - 2)(x - 3)(6x^2 + 7x + 2)$.

Step 4 (note): In addition to finding that 3 is a zero of P, we also found in the previous step that 3 is an upper bound for the zeros of $P(x)$, because the row that contains the quotient and the remainder has no negative entry. But, from Step 1, we already knew that there can not be more than two positive zeros. Had we not known from Descartes' Rule of Signs that we do not need to test further positive candidates, we still would not have needed to check for more, because all the remaining positive candidates are greater than 3.

Step 5: We continue by again factoring the quotient polynomial, which is now a quadratic.

$$
\begin{aligned}
P(x) &= (x-2)(x-3)(6x^2+7x+2) \\
     &= (x-2)(x-3)(2x+1)(3x+2) \quad \text{Factor quadratic}
\end{aligned}
$$

Therefore the zeros of P are 2, 3, $-\dfrac{1}{2}$, and $-\dfrac{2}{3}$.

Step 6: Finally, going back to the guidelines for graphing polynomial functions we discussed in Chapter 5 Section 1, we can graph

$$
\begin{aligned}
P(x) &= 6x^4 - 23x^3 + 3x^2 + 32x + 12 \\
     &= (x-2)(x-3)(2x+1)(3x+2).
\end{aligned}
$$

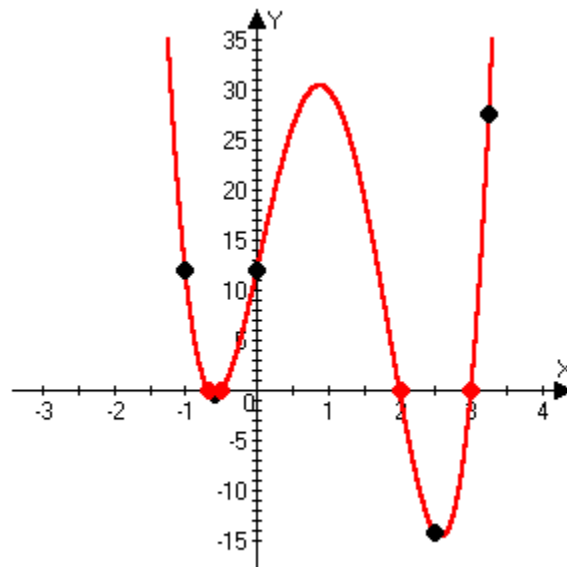We make our table of values using the zeros of P we found in the previous steps.

Example 5 (Continued):

x	$P(x)$
−1	12
−0.6667	0
−0.5833	−0.3857
−.5	0
0	12
2	0
2.5	−14.25
3	0
3.25	27.5391

P is of even degree and its leading coefficient is positive, therefore it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to -\infty$$

Thus, the graph of $P(x) = 6x^4 - 23x^3 + 3x^2 + 32x + 12$ is completed below.

$$P(x) = 6x^4 - 23x^3 + 3x^2 + 32x + 12$$

Example 6: Find all rational zeros of $P(x) = 2x^4 + 7x^3 + 9x^2 + 21x + 9$, and then find the irrational zeros, if any and graph the polynomial. If appropriate, use the Rational Zeros Theorem, the Upper and Lower Bounds Theorem, Descartes' Rule of Signs, the quadratic formula, or other factoring techniques.

Solution:

Step 1: Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros $P(x) = 2x^4 + 7x^3 + 9x^2 + 21x + 9$ can have.

$$P(x) = 2x^4 + 7x^3 + 9x^2 + 21x + 9$$
$$P(-x) = 2x^4 - 7x^3 + 9x^2 - 21x + 9$$

$P(x)$ has no positive real zeros, and zero, two or four negative real zeros, making a total of either zero, two or four real zeros.

Positive	Negative	Real
0	0	0
0	2	2
0	4	4

Step 2: Use the Rational Zeros Theorem to list the possible rational zeros of P, and synthetic division to begin testing candidates for zeros.

The possible rational zeros of P are

$$\pm\frac{1}{2}, \ \pm 1, \ \pm\frac{3}{2}, \ \pm 3, \ \pm\frac{9}{2}, \ \pm 9$$

Since in Step 1 we determined there are no positive real zeros, we can begin testing the negative candidates, starting with the value closest to zero.

$$
\begin{array}{r|rrrrr}
-\dfrac{1}{2} & 2 & 7 & 9 & 21 & 9 \\
& & -1 & -3 & -3 & -9 \\
\hline
& 2 & 6 & 6 & 18 & 0
\end{array}
\quad \leftarrow \ P\!\left(-\frac{1}{2}\right) = 0
$$

Example 6 (Continued):

So $-\dfrac{1}{2}$ is a zero, and

$$P(x) = \left(x + \dfrac{1}{2}\right)\left(2x^3 + 6x^2 + 6x + 18\right)$$

$$= (2x+1)\left(x^3 + 3x^2 + 3x + 9\right) \quad \textcolor{red}{\text{Factor 2 from last factor,}}$$

$$\textcolor{red}{\text{multiply into first factor}}$$

Step 3: Continue by factoring the quotient $x^3 + 3x^2 + 3x + 9$. Possible rational zeros are

$$\pm 1, \ \pm 3, \ \pm 9$$

Again we do not need to test the positive candidates because we know there are no positive real zeros. Starting with the value closest to zero,

$$
\begin{array}{r|rrrr}
-1 & 1 & 3 & 3 & 9 \\
   &   & -1 & -2 & -1 \\
\hline
   & 1 & 2 & 1 & 8 \\
\end{array}
\leftarrow \ \textcolor{blue}{-1 \text{ is not}}
$$
$$\textcolor{blue}{\text{a zero}}$$

$$
\begin{array}{r|rrrr}
-3 & 1 & 3 & 3 & 9 \\
   &   & -3 & 0 & -9 \\
\hline
   & 1 & 0 & 3 & 0 \\
\end{array}
\leftarrow \ \textcolor{blue}{P(-3) = 0}
$$

Thus -3 is a zero, and $P(x) = (2x + 1)(x + 3)(x^2 + 3)$.

Step 4: The quotient polynomial is a quadratic, which can be solved easily to give us two imaginary zeros $x = i\sqrt{3}$ and $x = -i\sqrt{3}$.
Therefore the real zeros of P are $-\dfrac{1}{2}$, and -3.

Step 5: Finally we will graph

$$P(x) = 2x^4 + 7x^3 + 9x^2 + 21x + 9$$

$$= (2x+1)(x+3)\left(x^2 + 3\right)$$

using the guidelines for graphing polynomial functions.

Example 6 (Continued):

x	$P(x)$
−3.25	18.6484
−3	0
−2	−21
−0.5	0
0	9

P has the following end behavior:

$$y \to \infty \;\; \text{as} \;\; x \to \infty \qquad \text{and} \qquad y \to \infty \;\; \text{as} \;\; x \to -\infty$$

$$P(x) = 2x^4 + 7x^3 + 9x^2 + 21x + 9$$
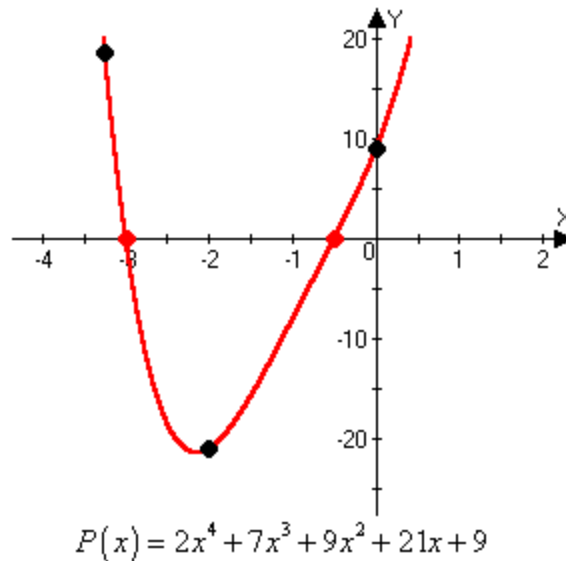
Up to this point we have studied polynomials in the real number system, and we have seen that an *n*th-degree polynomial can have at most *n* real zeros. In this section we will study polynomials in the complex number system where an *n*th-degree polynomial has exactly *n* zeros, and so can be factored into exactly *n* linear factors.

Fundamental Theorem of Algebra:

In 1799 the German mathematician C. F. Gauss proved the Fundamental Theorem of Algebra. This Theorem forms the basis for much of our work in factoring polynomials and solving polynomial equations.

Fundamental Theorem of Algebra:

Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ $\quad (n \geq 1, a_n \neq 0)$
with complex coefficients has at least one complex zero.

Because any real number is also a complex number, the theorem applies to polynomials with real coefficients as well.

The conclusion of the Fundamental Theorem of Algebra is that for every polynomial $P(x)$, there is a complex number c_1 such that

$$P(c_1) = 0.$$

From the Factor Theorem (Section 5.2), this tells us $x - c_1$ is a factor of $P(x)$. Thus we can write

$$P(x) = (x - c_1) Q(x)$$

where $Q(x)$ has degree $n - 1$. If the quotient $Q(x)$ has degree ≥ 1 we can repeat the procedure of obtaining a factor and a quotient with degree 1 less than the previous quotient until we arrive at the complete factorization of $P(x)$. This process is summarized by the next theorem.

Complete Factorization Theorem:

If $P(x)$ is a polynomial of degree $n > 0$, then there exist complex numbers a, c_1, c_2, . . . c_n (with $a \neq 0$) such that

$$P(x) = a(x - c_1)(x - c_2)...(x - c_n)$$

In the Complete Factorization Theorem the numbers c_1, c_2, . . . c_n are the zeros of P. These zeros need not all be different. If the factor $x - c$ appears k times in the complete factorization of $P(x)$, then we say that c is a zero of **multiplicity k**.

The next theorem follows from the Complete Factorization Theorem.

Zeros Theorem:

Every polynomial of degree $n \geq 1$ has exactly n zeros, provided that a zero of multiplicity k is counted k times.

Example 7:   Factor the polynomial $P(x) = 4x^5 - 324x$ completely and find all its zeros. State the multiplicity of each zero.

    Solution:

        Step 1:   Since $4x$ is a common factor, we have

$$P(x) = 4x^5 - 324x$$
$$= 4x(x^4 - 81)$$

        Step 2:   Next, using the formula for a difference of squares, we have

$$P(x) = 4x(x^4 - 81)$$
$$= 4x(x^2 - 9)(x^2 + 9)$$

        Step 3:   Again using the formula for a difference of squares, we have

$$P(x) = 4x(x^2 - 9)(x^2 + 9)$$
$$= 4x(x - 3)(x + 3)(x^2 + 9)$$

        Step 4:   To factor $x^2 + 9$, note that $3i$ and $-3i$ are zeros of this polynomial. Thus $x^2 + 9 = (x - 3i)(x + 3i)$, and so

$$P(x) = 4x(x - 3)(x + 3)(x^2 + 9)$$
$$= 4x(x - 3)(x + 3)(x - 3i)(x + 3i)$$

        Step 5:   Therefore the zeros of P are 0, 3, –3, $3i$ and $-3i$. Since each of the factors occurs only once, all the zeros are of multiplicity 1, and the total number of zeros is five.

The following table gives further examples of polynomials with their complete factorizations and zeros.

Degree	Polynomial	Zero(s)	Number of zeros
1	$P(x) = x - 3$	3	1
2	$P(x) = x^2 + 8x + 16$ $= (x+4)(x+4)$	-4 (multiplicity 2)	2
3	$P(x) = x^3 + x$ $= x(x-i)(x+i)$	$0, i, -i$	3
4	$P(x) = x^4 - 7x^2 - 144$ $= (x+3i)(x-3i)(x+4)(x-4)$	$-3i, 3i, -4, 4$	4
5	$P(x) = x^5 + 8x^3 + 16x$ $= x(x-2i)(x-2i)(x+2i)(x+2i)$	$0, 2i$ (multiplicity 2), $-2i$ (multiplicity 2)	5

Example 8: Find a polynomial with integer coefficients that satisfies the given conditions that P has degree 5, zeros 0, 2, $3i$, and $-3i$, with 2 a zero of multiplicity 2, and with $P(2) = 26$.

Solution:

Step 1: We are given that the zeros are 0, 2, $3i$, and $-3i$, with 2 a zero of multiplicity 2, thus by the factor theorem, the required polynomial has the form

$$P(x) = a(x-0)(x-2)^2(x-3i)\left(x-(-3i)\right)$$

Note that the factor $x - 2$ is squared because 2 is a zero with multiplicity 2.

Example 8 (Continued):

Step 2: Next we expand and simplify the polynomial.

$$P(x) = a(x-0)(x-2)^2(x-3i)(x-(-3i))$$
$$= ax(x-2)^2(x-3i)(x+3i)$$
$$= ax(x-2)^2(x^2+9) \qquad \text{Difference of squares}$$
$$= a(x^5 - 4x^4 + 11x^3 - 36x^2 + 18x) \qquad \text{Expand}$$

Step 3: Now we must determine what a is. We know that

$$P(2) = a(2^5 - 4\cdot2^4 + 11\cdot2^3 - 36\cdot2^2 + 18\cdot2)$$
$$= -52a$$
$$= 26$$

So $a = -\frac{1}{2}$.

(Note: If we are given no information about P other than its zeros and their multiplicity, we can choose any number for a. Usually $a = 1$ is the simplest choice.)

Step 4: Thus

$$P(x) = \tfrac{1}{2}(x^5 - 4x^4 + 11x^3 - 36x^2 + 18x)$$
$$= \tfrac{1}{2}x^5 - 2x^4 + \tfrac{11}{2}x^3 - 18x^2 + 9x$$

Example 9: Find all zeros of the polynomial $P(x) = x^4 + x^3 + 3x^2 - 5x$.

Solution:

Step 1: We begin by factoring the polynomial $P(x) = x^4 + x^3 + 3x^2 - 5x$ using the tools we studied in previous sections. Since x is a common factor, we have

$$P(x) = x^4 + x^3 + 3x^2 - 5x$$
$$= x(x^3 + x^2 + 3x - 5) \qquad \text{Factor } x$$

Example 9 (Continued):

From the Rational Zeros Theorem (Section 5.3), we obtain the following list of possible rational zeros: $\pm 1, \pm 5$. Checking these using synthetic division, we find that 1 is a zero and we get the following factorization.

$$P(x) = x\left(x^3 + x^2 + 3x - 5\right)$$
$$= x(x-1)\left(x^2 + 2x + 5\right) \qquad \text{Factor } x - 1$$

Step 2: Using the quadratic equation, the zeros of the quadratic factor are

$$x = \frac{-2 \pm \sqrt{4 - 20}}{2}$$
$$= -1 \pm 2i$$

Step 3: Thus the zeros of $P(x)$ are 0, 1, $-1 + i$, and $-1 - i$.

As you may have noticed from the examples so far, the complex zeros of polynomials with real coefficients come in pairs. Whenever $a + bi$ is a zero, its complex conjugate $a - bi$ is also a zero.

Conjugate Zeros Theorem:

If the polynomial P has real coefficients, and if the complex number z is a zero of P, then its complex conjugate $\bar{z}$ is also a zero of P.

Example 10: Find a polynomial with integer coefficients that satisfies the given conditions that P has degree 4, and zeros -3, ½, and $1 + i$.

Solution:

Step 1: In this problem we are instructed to find a polynomial of degree 4, but we are only given three zeros, all of multiplicity 1. But, since $1 + i$ is a zero, we know $1 - i$ by the Conjugate Roots Theorem, and so we have a fourth zero.

Step 2: The zeros of $P(x)$ are -3, ½ , $1 + i$, and $1 - i$, so the required polynomial has the form

$$P(x) = a\left(x - (-3)\right)\left(x - \tfrac{1}{2}\right)\left(x - (1+i)\right)\left(x - (1-i)\right)$$

Example 10 (Continued):

    Step 3: Now we expand and simplify the equation.

$$\begin{aligned}
P(x) &= a\left(x-(-3)\right)\left(x-\tfrac{1}{2}\right)\left(x-(1+i)\right)\left(x-(1-i)\right) \\
&= a(x+3)\left(x-\tfrac{1}{2}\right)(x-1-i)(x-1+i) \\
&= a\left(x^2+\tfrac{5}{2}x-\tfrac{3}{2}\right)(x-1-i)(x-1+i) && \text{Expand} \\
&= a\left(x^2+\tfrac{5}{2}x-\tfrac{3}{2}\right)\left((x-1)-i\right)\left((x-1)+i\right) && \text{Regroup} \\
&= a\left(x^2+\tfrac{5}{2}x-\tfrac{3}{2}\right)\left((x-1)^2-i^2\right) && \text{Difference of Squares} \\
&= a\left(x^2+\tfrac{5}{2}x-\tfrac{3}{2}\right)\left(x^2-2x+2\right) && \text{Expand} \\
&= a\left(x^4+\tfrac{1}{2}x^3-3x^2+\tfrac{13}{2}x-2\right) && \text{Expand}
\end{aligned}$$

    Step 4: Finally we must make all coefficients integers. The lowest common denominator of the coefficients is 2, so we set $a = 2$ and get

$$\begin{aligned}
P(x) &= 2\left(x^4+\tfrac{1}{2}x^3-3x^2+\tfrac{13}{2}x-2\right) \\
&= 2x^4+x^3-6x^2+13x-4
\end{aligned}$$

    Note that any other polynomial that satisfies the given requirements must be an integer multiple of this one.

We have seen that a polynomial factors completely into linear factors if we use complex numbers. If we do not use complex numbers, then a polynomial with real coefficients can always be factored into linear and quadratic factors. A quadratic polynomial with no real zeros is called **irreducible** over the real numbers. Such a polynomial cannot be factored without using complex numbers.

    Linear and Quadratic Factors Theorem:

    Every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors with real coefficients.

Example 11: Given the polynomial $P(x) = x^4 + 2x^2 - 63$

(a) Factor P into linear and irreducible quadratic factors with real coefficients.
(b) Factor P completely into linear factors with complex coefficients.

Solution (a):

$$\begin{aligned}
P(x) &= x^4 + 2x^2 - 63 \\
&= \left(x^2 - 7\right)\left(x^2 + 9\right) && \text{\color{red}Factor like a quadratic} \\
&= \left(x - \sqrt{7}\right)\left(x + \sqrt{7}\right)\left(x^2 + 9\right) && \text{\color{red}Quadratic Equation} \\
& && \text{\color{red}or difference of squares formula}
\end{aligned}$$

The factor $x^2 - 9$ is irreducible since it has only the imaginary zeros $\pm\, 3i$.

Solution (b):

$$\begin{aligned}
P(x) &= \left(x - \sqrt{7}\right)\left(x + \sqrt{7}\right)\left(x^2 + 9\right) \\
&= \left(x - \sqrt{7}\right)\left(x + \sqrt{7}\right)\left(x - 3i\right)\left(x + 3i\right) && \text{\color{red}Quadratic Equation}
\end{aligned}$$