Chapter 5

Transforming Shapes

It is difficult to walk through daily life without being able to see geometric transformations in your surroundings. Notice how the leaves of plants, for example, are almost a mirror image of themselves across their center. Also take note of how the leaves get smaller as they travel up the plant while still keeping their similar shape. Both of these traits are examples of geometric transformations. What else shows these traits in nature?

During This Chapter

- You will investigate translations, reflections, rotations, and dilations of figures both on and off the coordinate plane.
- You will apply multiple transformations to a preimage and identify the transformations applied to an image.
- You will investigate and identify the types and properties of symmetry.

Application

You use transformations in space all the time without realizing it. The Earth moves through an elliptical orbit that is described by a rotation. A school bus moving down the street is changing its position without altering its other properties, an example of a rigid transformation. In this section, you will learn how to describe transformations algebraically and determine their geometric characteristics.

Section 5.1

Translations

Objectives

- Determine the congruence of a preimage and an image under a translation
- Use coordinate function notation to describe and apply translations on the coordinate plane
- Use prime notation to denote transformation points

New Vocabulary

- Transformation
- Preimage
- Image
- Prime notation
- Translation
- Translation vector
- Coordinate function notation
- Rigid transformation

Although regulation basketball hoops are 10 ft off the ground, many have the ability to have their height lowered in order to accommodate younger players or beginners just learning the skills. When the hoop is lowered, nothing changes other than the location off the ground. The backboard, rim diameter, and net remain the same size and dimensions. Imagine if a similar movement occurred on a coordinate plane. How can this shift be described?

Translation

Look at $\triangle ABC$ in **Figure 5.1-1**. Notice that next to it is a copy of itself. Because it is in a different location, the second triangle is not exactly the same as the first one. It is a transformation of the original triangle. A transformation is a function that maps a set to another set or itself. These modifications can include changes in the figure's position, orientation, shape, or size.

The preimage of a transformation is the original figure before it is transformed. In this example, $\triangle ABC$ is the preimage. The image is the figure after it has been transformed. In this example, $\triangle A'B'C'$ is the image. Note that point A' (pronounced "A prime") corresponds to point A, B' corresponds to B, and C' corresponds to C. This system of naming transformed points by following their label with the prime symbol is called prime notation. If the image is transformed again, then a double prime is used to label each point in the second image. For example, A in the preimage becomes A' in the image, and that then becomes A'' in the second image. The second transformation would result in $\triangle A''B''C''$.

In this example, $\triangle ABC$ has been moved to a different location without any change in its size or orientation. A translation occurs when every point on an object is shifted by the same distance and in the same direction. $\triangle ABC$ has been translated because each point in the preimage has been shifted up and to the right by the same amount. The result is an image that is congruent to the preimage and in the same orientation.

The process of translation is shown in **Animation 5.1-1**. Note that the entire shape moves as a unit, so every part of it moves the same distance and direction. The distance and direction by which a figure is translated is known as the translation vector. The translation vector can be represented as an arrow, as it is for the shape in **Animation 5.1-1**. Every part of the object is shifted by the distance and in the direction shown by the arrow.

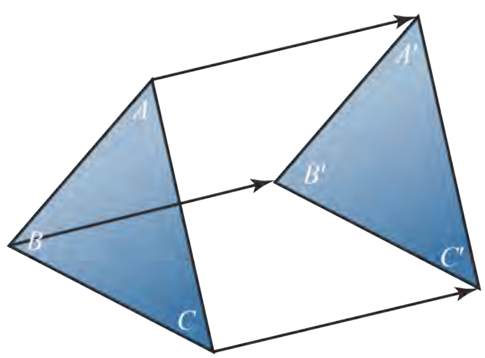
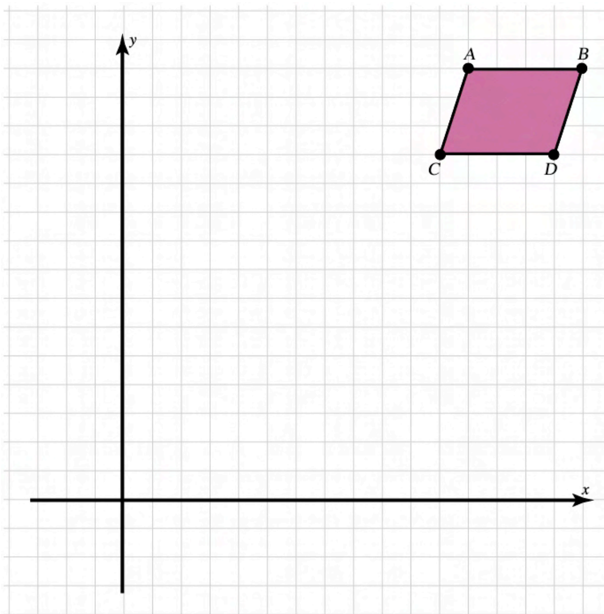


Figure 5.1-1 A transformation has been made to triangle *ABC*, changing its position.



Animation 5.1-1 In a translation, every part of the preimage is shifted the same distance and direction.

Translation on the Coordinate Plane

Any translation on the coordinate plane can be represented as a shift in the x-direction and a shift in the y-direction. **Figure 5.1-2** shows $\triangle DEF$ and its image, which is a translation of the preimage. Note that each point on the triangle has been translated three units to the right and two units up. In terms of coordinates, the image was created by adding 3 to each x-value and 2 to each y-value of the preimage.

One way to specify that $\triangle DEF$ should be translated by this vector is by writing a function that describes the operations that should be done on x and y for each point in the figure. <u>Coordinate function notation</u> is a way of describing in ordered pair form the operations that should be done on each point in order to transform a figure. The translation of $\triangle DEF$ three units to the right and two units up can be written in coordinate function notation as $(x, y) \rightarrow (x+3, y+2)$. This notation states that adding 3 to each x-value and 2 to each y-value of the preimage will produce the desired image.

Some types of transformations produce translations, and others do not. The transformation $(x, y) \rightarrow (2x, 2y)$ is shown in **Figure 5.1-3** for $\square GHJK$. Notice that point G does not move, but the other points move away from the origin. The result is a square twice as large as the preimage. For the following reasons, this transformation is not a translation: the vertices do not all follow the same vector, and the image is not the same size as the preimage.

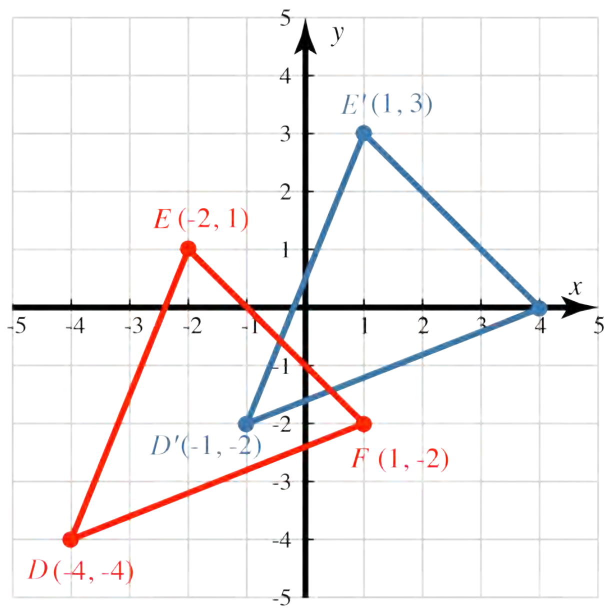


Figure 5.1-2 The triangle is translated three units to the right and two units up.

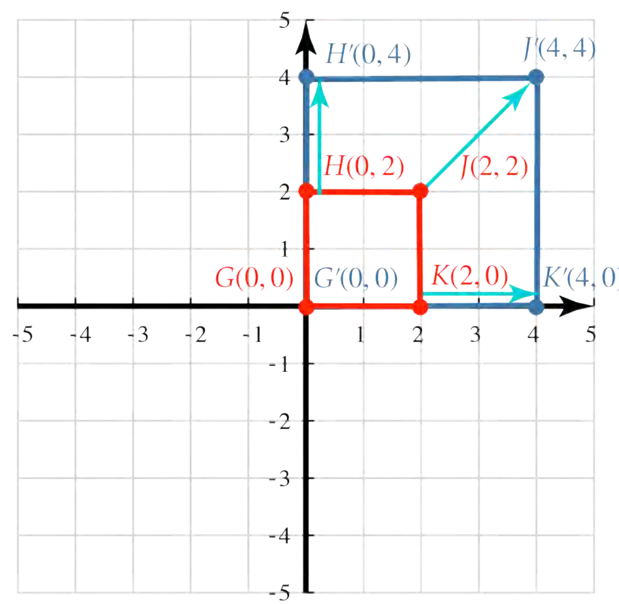


Figure 5.1-3 Because the vertices are shifted by different amounts and in different directions, this transformation is not a translation.

Using Coordinate Function Notation to Determine Translations

The transformation $(x, y) \rightarrow (-x, -y)$, shown for $\triangle LMN$ in **Figure 5.1-4**, is not a translation either. Although the image is the same size as the preimage, it is in a different orientation. Each vertex is shifted down and to the left, but they are shifted by different distances.

The transformation shown in **Figure 5.1-5** could be produced by a translation. Each vertex in the preimage is shifted the same distance and direction, as shown by the arrows. Notice that the arrows are all the same length and point in the same direction.

Since translating a figure involves shifting each point the same way both horizontally and vertically, any translation can be described by the function $(x, y) \rightarrow (x+h, y+k)$, where h and k are the horizontal and vertical components of the translation vector. That is, the figure is shifted h units to the right and k units upward. A negative value for h indicates a leftward shift, and a negative value for k indicates a downward shift.

For example, consider the rectangle shown in **Figure 5.1-6**. It has undergone the translation $(x, y) \rightarrow (x+3, y-2)$. Notice that shifting the bottom-left vertex from T to T' required adding 3 to the x-value and subtracting 2 from the y-value. Similarly, for each point (x, y) on the preimage, the corresponding point on the image is $(x+3, y-2)$.

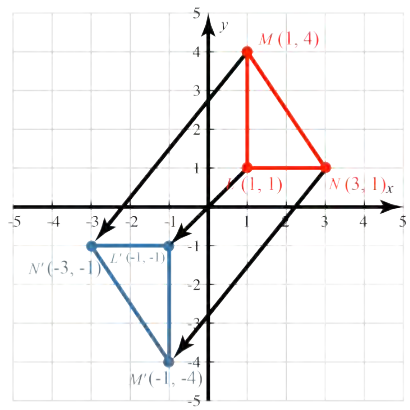


Figure 5.1-4 Because the vertices are shifted by different amounts, this transformation is not a translation.

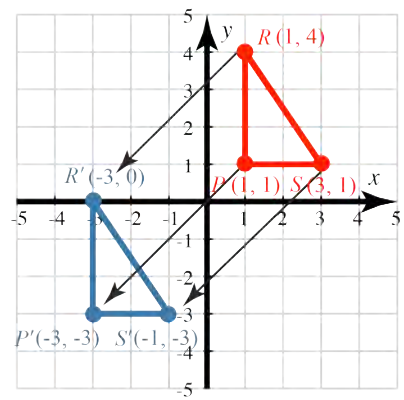


Figure 5.1-5 This transformation could be produced by a translation.

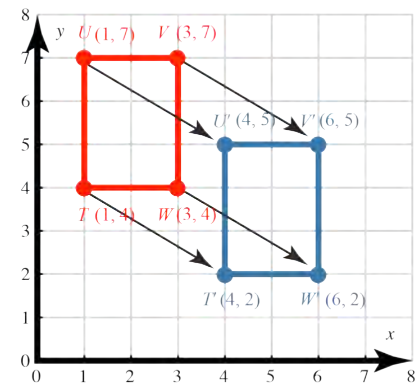
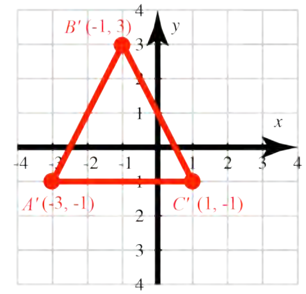


Figure 5.1-6 For any point (x, y) on the preimage, the corresponding point on the image is $(x+3, y-2)$.

Example problem — Draw the preimage used to create $\triangle A'B'C'$ by the transformation $(x, y) \to (x - 2, y + 1)$.



Analyze — The problem gives an image of a triangle as well as the transformation used to produce the image. It asks for a drawing of the preimage.

Formulate — The question asks for the original coordinates of the figure before the translation. First, the effects of the transformation by using the inverse operation of the one provided in the coordinate function notation of the transform. For each vertex of the image, add 2 to its x-coordinate and subtract 1 from its y-coordinate to find the corresponding coordinate of the preimage.

Determine —

$A(-3 + 2, -1 - 1)$ — Add 2 to the x-coordinate, and subtract 1 from the y-coordinate.

$A(-1, -2)$

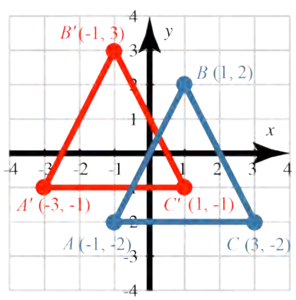
$B(-1 + 2, 3 - 1)$ — Add 2 to the x-coordinate, and subtract 1 from the y-coordinate.

$B(1, 2)$

$C(1 + 2, -1 - 1)$ — Add 2 to the x-coordinate, and subtract 1 from the y-coordinate.

$C(3, -2)$

Graph triangle ABC on a coordinate plane.

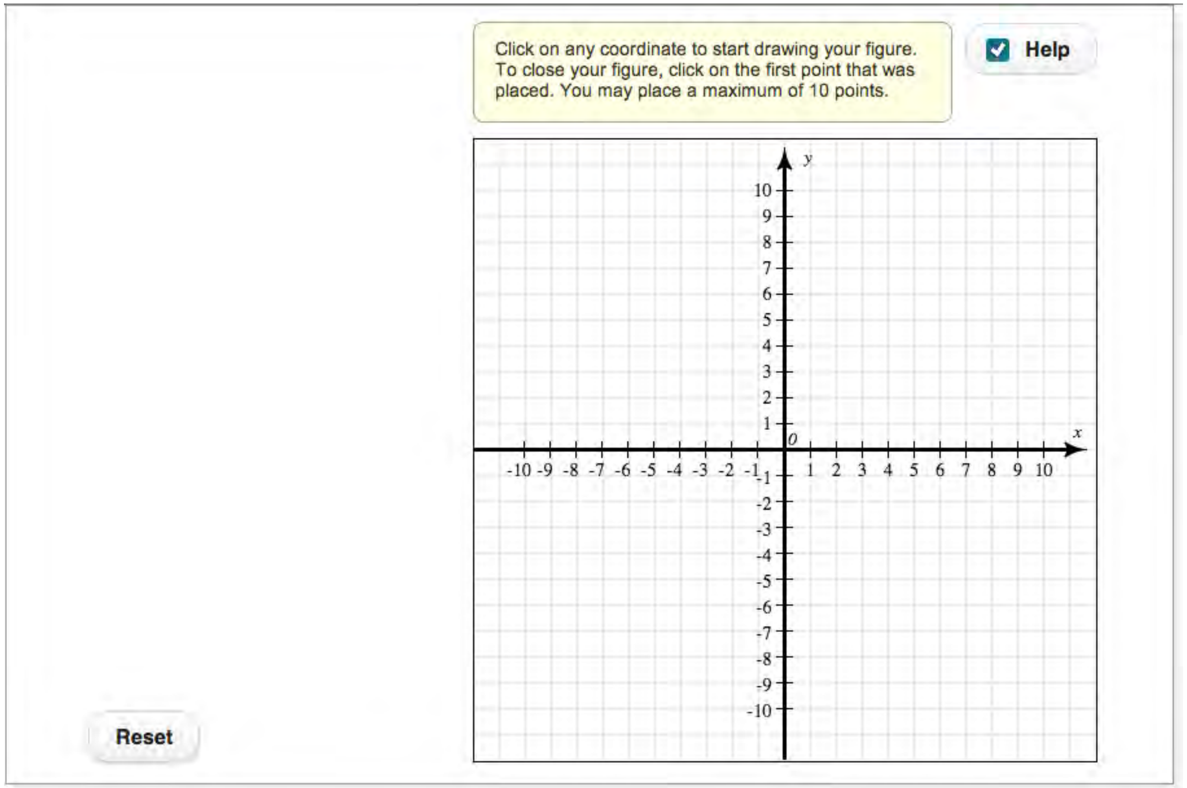


Justify — Because the image is formed by subtracting 2 from each x-coordinate and adding 1 to each y-coordinate of the preimage, the reverse operations were followed to obtain the preimage from the image.

Evaluate — Reversing the operations from the transformation function provided an efficient way to obtain the preimage. The answer is reasonable because the image is congruent to the preimage.

Virtual Manipulative - Image Translation

Any translation on the coordinate plane can be represented as a shift in the x-direction and a shift in the y-direction. In this virtual manipulative, you will create an image and use coordinate function notation to translate it across the coordinate plane by adjusting the notation or by dragging.



You should have been able to translate a shape across a coordinate plane by adjusting the coordinate function notation or by dragging. Notice how, no matter where the image is being translated, the distance between each preimage point and image point are equal and remain so with every translation.

Properties of Translations

Because the process of translation involves shifting every point on a figure the same distance and direction, it produces an image that has the same size, shape, and orientation as the preimage. A rigid transformation is a transformation in which the image is congruent to the preimage. A translation is one type of rigid transformation.

Not every rigid transformation is a translation. **Figure 5.1-7** shows an image that is congruent to its preimage but is in a different orientation. This is a rigid transformation because the figures have the same shape and size. However, because the image and preimage have different orientations, it is not a translation.

Look at the figure and its translation shown in **Figure 5.1-8**. Because translations produce congruent images, $\triangle XYZ$ and $\triangle X'Y'Z'$ are congruent. However, it is not only the triangles that are congruent but also their individual parts. $\overline{X'Y'}$ is a translation of $\overline{XY}$, so these segments are congruent. Similarly, $\overline{Y'Z'}$ is congruent to $\overline{YZ}$, and $\overline{X'Z'}$ is congruent to $\overline{XZ}$.

Jump to Distance in the Coordinate Plane

Not only are the segments in the image congruent to the segments in the preimage, but the angles are also congruent. For example, $\angle X'$ is congruent to $\angle X$. It is possible to identify the congruent parts as a result of the prime notation even without looking at the figure because the prime notation identifies the corresponding parts of the two figures. Corresponding parts of congruent figures are always congruent.

Jump to Congruent Angles

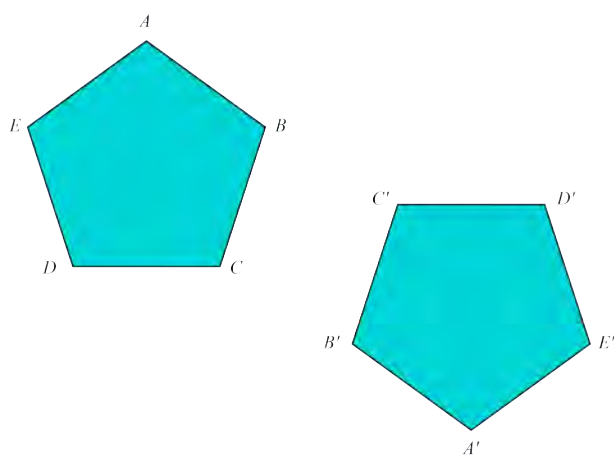


Figure 5.1-7 This image is a rigid transformation, but not a translation, of the preimage.

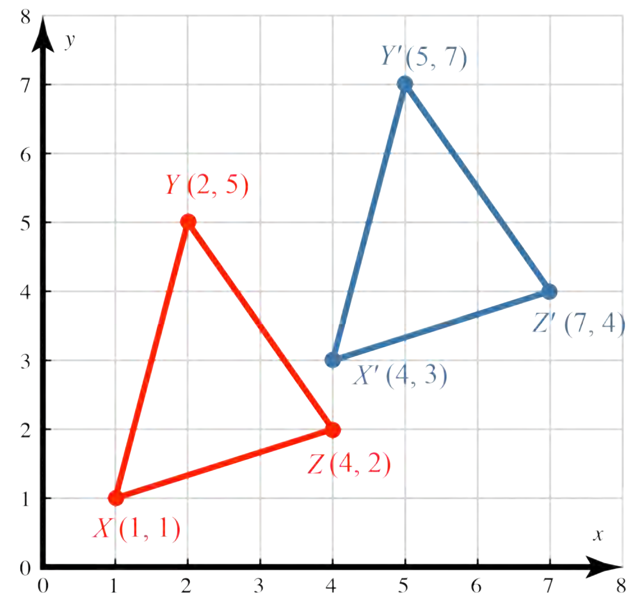
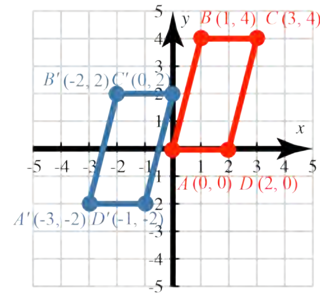


Figure 5.1-8 Each part of triangle $X'Y'Z'$ is congruent to the corresponding part of triangle XYZ.

5.1 - Translations



Example problem Determine whether $\overline{CD}$ and $\overline{C'D'}$ are parallel.

Analyze The problem shows a preimage and an image produced by a transformation. It asks whether $\overline{C'D'}$ on the image is parallel to $\overline{CD}$ on the preimage.

Formulate Calculate the slope of $\overline{CD}$ and of $\overline{C'D'}$. Compare the slopes to see whether they are the same.

Determine

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Write the slope formula.

$$m_{CD} = \frac{0 - 4}{2 - 3}$$ Plug in coordinates from points C and D.

$$m_{CD} = \frac{-4}{-1}$$ Simplify.

$$m_{CD} = 4$$

$$m_{C'D'} = \frac{-2 - 2}{-1 - 0}$$ Plug in coordinates from points C and D.

$$m_{C'D'} = \frac{-4}{-1}$$ Simplify.

$$m_{C'D'} = 4$$

$$m_{CD} = m_{C'D'}$$ Compare the two slopes.

Justify Lines or segments are parallel if and only if they have the same slope. Segments $\overline{CD}$ and $\overline{C'D'}$ have the same slope, so they are parallel.

Evaluate Calculating the slopes provided a precise method for determining whether the segments were parallel. The calculation confirmed the similar appearance of the slopes on the diagram.

Section 5.2

Reflections

Objectives

- Determine the congruence of a preimage and image under a reflection
- Apply reflections in the coordinate plane using a line of reflection
- Use coordinate function notation to describe and apply reflections across the *x*- and *y*-axis

New Vocabulary

- Reflection
- Line of reflection

When we wake up in the morning and begin to get ready for school, most of us will take a glance in the mirror. Your reflection in the mirror appears to be the same distance away from the mirror as you are. If you look at your reflection as if it were a person looking at you, which hand does the person appear to use? How would you create a reflection? What if the mirror were a line?

Reflection Across a Line

A reflection is a transformation in which one object is a mirror image of another or, in other words, in which an object has been flipped over a certain line. The original geometric shape is called a preimage, and the reflected image is the image or reflection. If a perpendicular line is drawn from each vertex of a preimage to the line over which it is being reflected, each vertex in the image will be the same distance away on that same perpendicular. This is true for every point in the preimage, but mapping the vertices simplifies determining these points in the image.

Any line can be used to flip the original shape over, and it is the line that acts as the "mirror." In **Figure 5.2-1** the original triangle, $\triangle ABC$, has been reflected over the y-axis, creating $\triangle A'B'C'$. Each point has been moved perpendicularly across the y-axis so that it is the same distance from the line on the other side. In this way, the y-axis is the mirror, or line of reflection. A line of reflection is the line that an object is reflected over. This means that a point that has been reflected is the mirror image on the other side of the line of reflection.

A line of reflection does not have to be a vertical or horizontal line. In **Figure 5.2-2,** a trapezoid has been reflected over the provided line. From each original vertex, a perpendicular line has been drawn to the line of reflection, and that distance has been measured. At the same distance on the opposite side of the line of reflection is the reflected point.

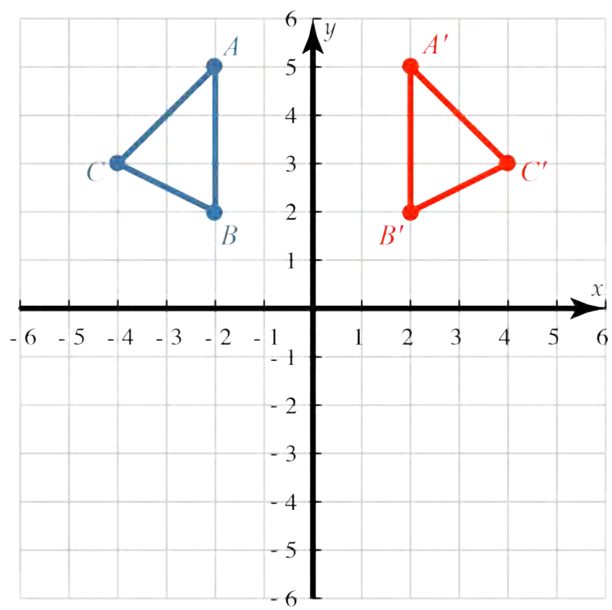


Figure 5.2-1 The triangle in quadrant I is the reflected image of the triangle in quadrant II.

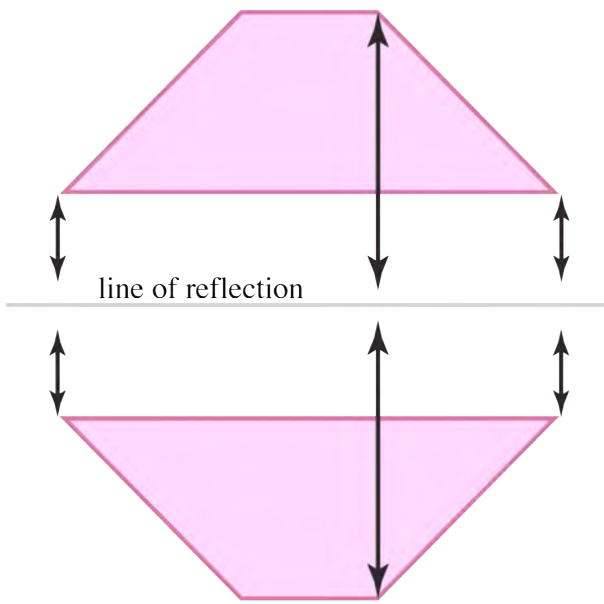


Figure 5.2-2 A line of reflection can be in any direction.

When a Point Falls on the Line of Reflection

There is a special circumstance should a point fall on the line of reflection. Imagine holding a penny above a piece of metal that has a mirror finish on it. A reflection of the penny is seen until the penny sits on the mirror. At that point, the reflection cannot be seen. The same is true with a line of reflection. A point that is located on the line of reflection will not appear to have a reflection because the original point is getting in the way. Another way to think of this is that the point and the reflection are the same.

All other points besides those directly on the line of reflection will be moved perpendicularly across the line of reflection the same distance away from the line as the preimage. **Figure 5.2-3** illustrates this case: two points, *B* and *C*, are not on the *x*-axis, which is acting as the line of reflection, and one point, *A*, is directly on the *x*-axis. The two points not on the *x*-axis have a reflection different from the original point in the preimage, whereas the point sitting directly on the *x*-axis acts as its own reflection.

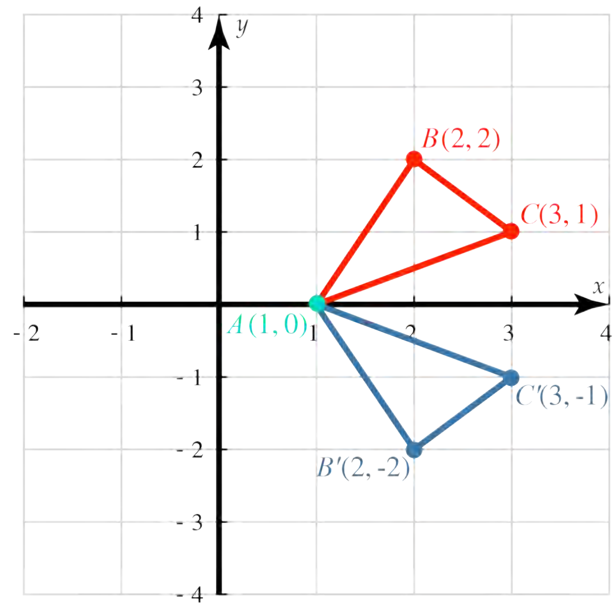


Figure 5.2-3 If a point is on the actual line of reflection, it will not move during a reflection transformation.

Reflection in the Coordinate Plane

When reflecting across a line in the coordinate plane, special attention can be paid to the coordinates and the relationship between the coordinates of the preimage and image. More specifically, a relationship exists between the coordinates of the preimage and image. If the y-axis is the line of reflection, as it is in **Figure 5.2-1**, the y-coordinate of the new point remains the same, but the x-coordinate of the new point changes to the opposite sign.

In **Figure 5.2-4**, the preimage is □ABCD. The coordinates for each vertex are given. Suppose the line of reflection is not the y-axis but is instead the line $x = 3$. Just as when reflecting over the y-axis, the y-coordinates will remain the same, but the x-coordinates will need to move the same distance on the other side of the line $x = 3$. Thus, the vertex $(-1, 0)$ will become $(7, 0)$. The y-coordinate, 0, remains the same. Since the point $(-1, 0)$ is four units from the line $x = 3$, it will need to move four units to the right of that line to $(7, 0)$. The point $(-2, 0)$ is five units to the left of the line $x = 3$. Moving it five units to the right will make it become $(8, 0)$. Using this same process, $(-2, 1)$ becomes $(8, 1)$, and $(-1, 1)$ becomes $(7, 1)$.

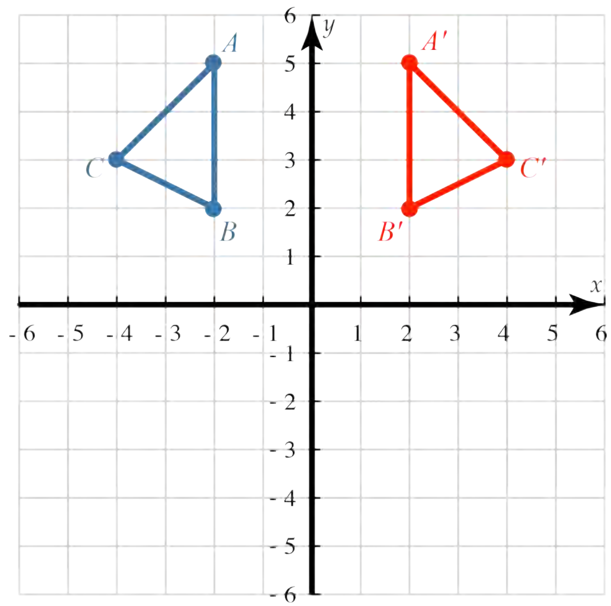


Figure 5.2-1 The triangle in quadrant I is the reflected image of the triangle in quadrant II.

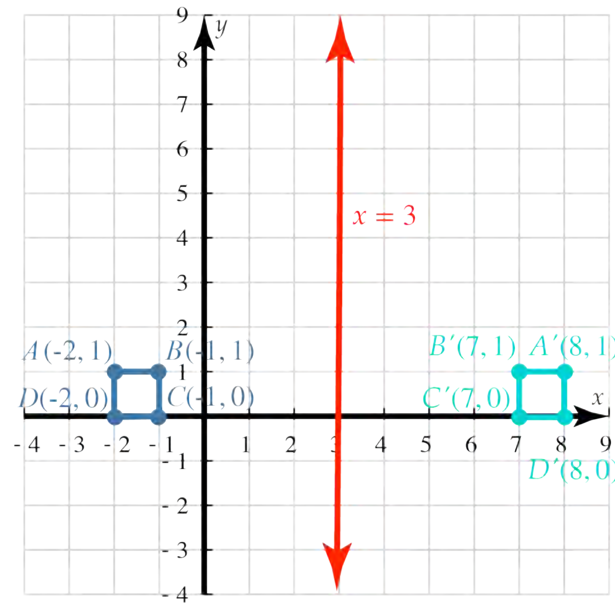
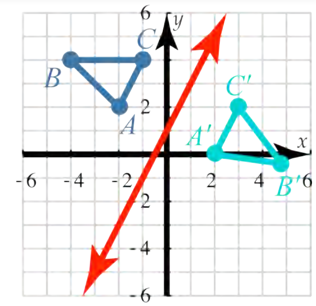


Figure 5.2-4 The reflection of a figure over a vertical line will keep the same y-coordinates, but the x-coordinates will change based upon their distance from the line of reflection.



Example problem

Given $\triangle ABC$ and its reflection $\triangle A'B'C'$, determine the equation of the line of reflection.

Analyze

The problem says to find the equation of the line for the line of reflection given a preimage and image.

Formulate

Because the line of reflection is halfway between the preimage and image, find the midpoints of the segments joining the corresponding points. Then, using two of these points, find the slope and then the y-intercept in order to get the equation of the line.

Determine

Pick two pairs of corresponding points: (–2,2) and (2,0) and (–1,4) and (3,2).

$\frac{-2+2}{2} = 0$ and $\frac{2+0}{2} = 1$, so this midpoint is (0,1) and is located on the y-axis.

$\frac{-1+3}{2} = 1$ and $\frac{4+2}{2} = 3$, so this midpoint is (1,3).

Finding the slope yields $\frac{3-1}{1-0} = 2$.

Justify

Plugging the slope into the slope-intercept form gives the equation $y = 2x + 1$. This process is verified by examining the y-intercept of the graph and checking the slope between two points on the graph by counting up and over between the two points.

Evaluate

Using the midpoints of the segments connecting the vertices from the pre-image to the image gave a reasonable way to determine the equation of the line. If the reflection had been a horizontal or vertical reflection, only one of the midpoints would have been necessary because the slope would already have been known.

Using Coordinate Notation to Describe Reflections

When reflecting across any axis, coordinate notation is used to describe what the reflection does to the vertices. For the reflection over the y-axis, $P(x,y) \rightarrow P'(-x,y)$. This notation shows that the y-coordinates remain the same and the x-coordinate becomes the opposite. If the x-axis is the line of reflection, the x-coordinate of the new point remains the same, but the y-coordinate of the new points becomes opposite in sign, $P(x,y) \rightarrow P'(x,-y)$, as illustrated in **Figure 5.2-5**.

Likewise, if a figure is reflected across the line $y = x$, then the x- and y-coordinates will switch places, giving $P(x,y) \rightarrow P'(y,x)$, as illustrated in **Figure 5.2-6**.

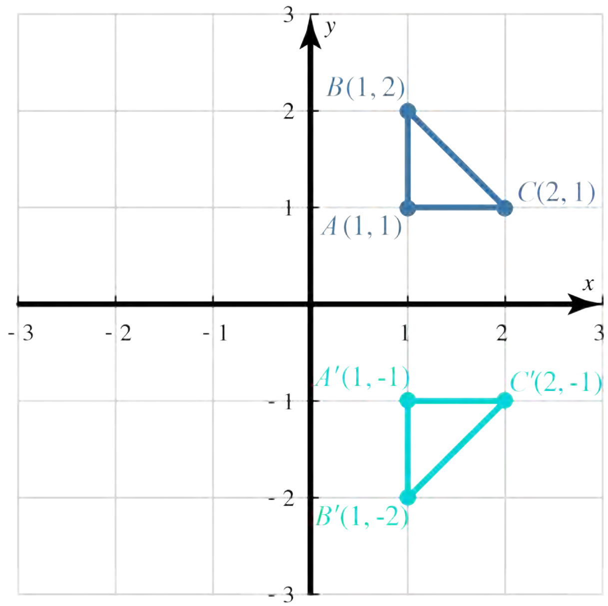


Figure 5.2-5 If an object is reflected across the *x*-axis, then the *x*-coordinate remains the same and the *y*-coordinate becomes its opposite.

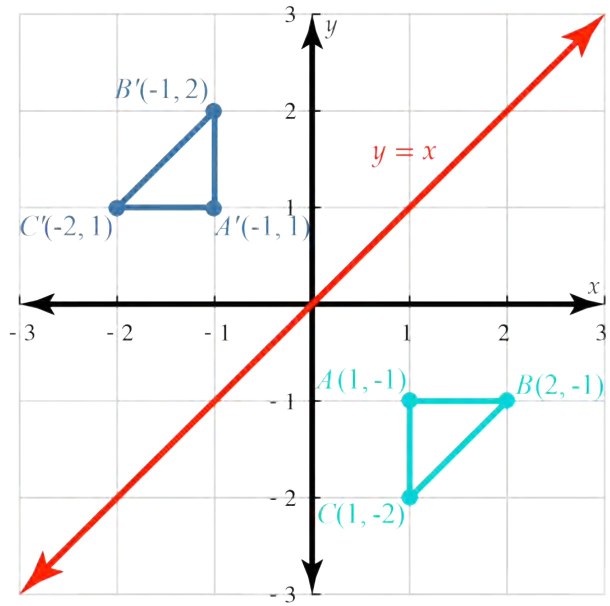
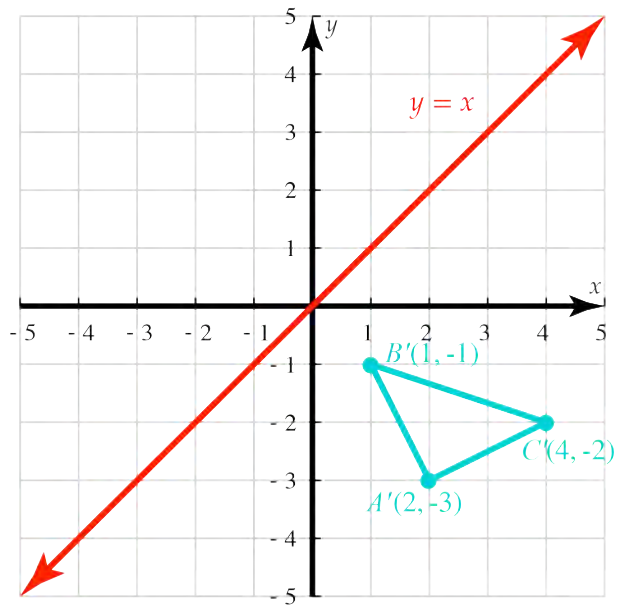


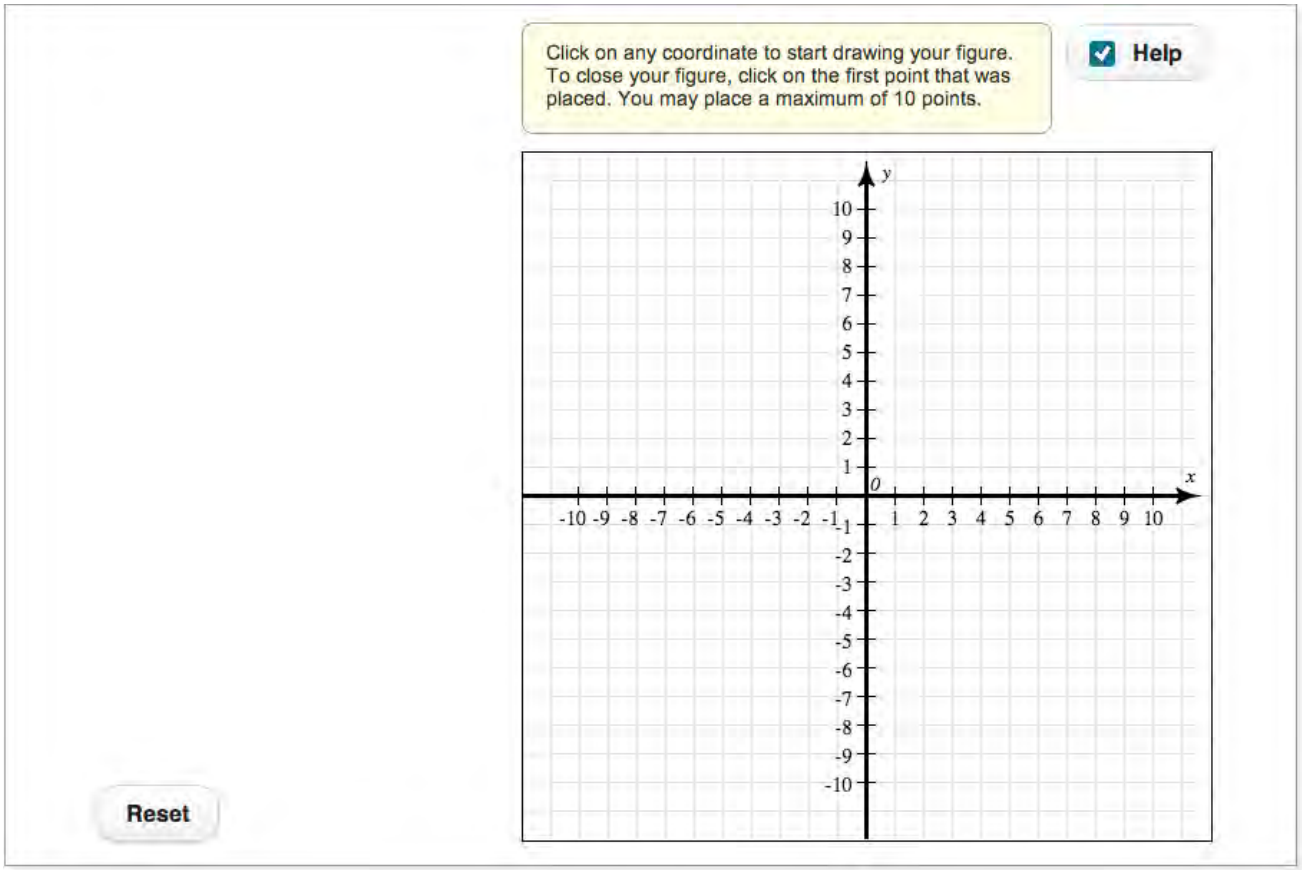
Figure 5.2-6 If an object is reflected over the line *y* = *x*, then the *x*- and *y*-coordinates will switch places in the reflection.

Example problem	$\triangle ABC$ was reflected across the line $y = x$. Given $\triangle A'B'C'$, find the coordinates for $\triangle ABC$.
Analyze	The problem gives the reflection. If the line of reflection is known to be $y = x$, find the coordinates for the preimage.
Formulate	Because reflecting over the line $y = x$ swaps the x- and y-coordinates, to get the coordinates for the preimage, each x- and y-coordinate must be switched.
Determine	$(2, -3) \rightarrow (-3, 2)$ $(1, -1) \rightarrow (-1, 1)$ $(4, -2) \rightarrow (-2, 4)$
Justify	The coordinates for the preimages are $A(-3,2)$, $B(-1,1)$, and $C(-2,4)$. This process is verified by switching the x- and y-coordinates for each vertex in the image.
Evaluate	Knowing the rules about what happens to coordinates when they are reflected over $y = x$, the x-axis, and the y-axis helps with this problem. Plotting the points of $\triangle ABC$ demonstrates reasonableness.



Virtual Manipulative - Image Reflection

When reflecting an image across axes on a coordinate plane, the relationship between the coordinates of the preimage and image can be seen in the coordinate notation. In this virtual manipulative, you will be able to draw a geometric image and reflect it over the x- and y-axis.



As you applied reflections of your drawn image, you should have been able to see how the coordinate notation reflected those changes. And while you only able to reflect over the x- and y-axis, you should have discovered that reflecting over both axes at the same time allowed you to essentially reflect over the function $y = x$.

Properties of Reflections

Notice that for each preimage and image in a reflection, the figures were the same size and shape. The figures, if possible, could still be folded over on top of one another for a perfect fit. Thus, the only change that occurred was the orientation and location of the figure. This type of transformation is considered a rigid transformation or isometry, which means the preimage and image are congruent. Other transformations that fall under the category of rigid transformations include rotations and translations, making these transformations isometries as well. A reflection is referred to as an opposite isometry because the points of the image are in the opposite order of those in the preimage.

Because of this congruence, it should be noted specifically what is maintained between the image and the preimage. Distance is preserved, meaning the distance between points remains the same. Also, all angle measures remain the same. Parallelism is maintained; that is, if two segments were parallel in the preimage, they will remain parallel in the image. Collinearity is also maintained in that points remain on the same lines. Finally, midpoints are maintained in that midpoints in the preimage are mapped to the midpoints in the image. What is not preserved is the orientation. Because of the nature of a reflection, the points will be in opposite order, as illustrated in **Figure 5.2-7**.

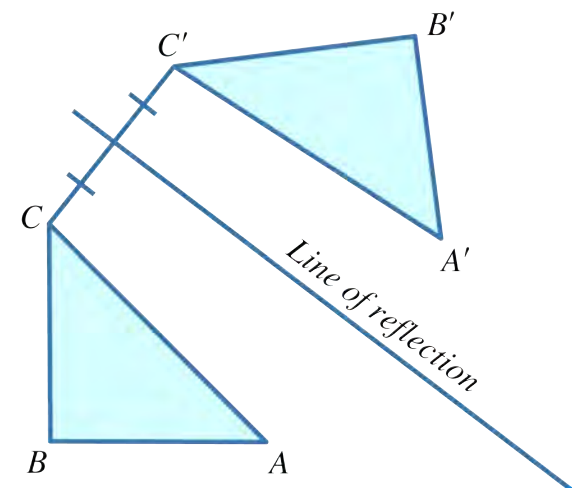


Figure 5.2-7 Reflections maintain many properties; however, orientation is not being maintained causing the vertices to appear in opposite order.

Section 5.3

Rotations

Objectives

- Determine the congruence of a preimage and image under a rotation
- Use a ruler and protractor to construct rotations
- Use coordinate function notation to describe and apply rotations on the coordinate plane

New Vocabulary

- Rotation
- Center of rotation
- Angle of rotation

As an analog clock runs, the hands constantly move, changing the directions in which they point. In the course of one minute, the second hand rotates 360° and faces all possible directions in the plane. The words “clockwise” and “counterclockwise,” which are used to describe the direction of rotation, come from the motion of clock hands. Do the shape and size of the hands change as they move? Is there any point on the hands that does not move?

Rotation About a Point

A rotation is a transformation in which each point on a preimage except the center turns through a specified angle around a certain point while maintaining the same distance from that point. The point around which the preimage rotates is called the center of rotation. The angle through which it rotates is called the angle of rotation.

Figure 5.3-1 shows the rotation of a shape. The blue arrow shows the path that the shape travels as it rotates. The center of rotation is at the intersection of the red segments, and the angle made by this intersection is the angle of rotation. Notice that each point on the image is the same distance from the center of rotation as the corresponding point on the preimage.

Objects can be rotated in different directions. Suppose that a problem asks for triangle *ABC*, shown in **Figure 5.3-2**, to be rotated by 90°. The preimage can be rotated either clockwise or counterclockwise. Rotating 90° clockwise produces image $A'B'C'$, and rotating 90° counterclockwise produces image $A''B''C''$. Angles in math are usually measured counterclockwise, so if a direction is not specified, the rotation is assumed to be counterclockwise. Triangle $A'B'C'$ can be considered either a 270° rotation or a 90° clockwise rotation.

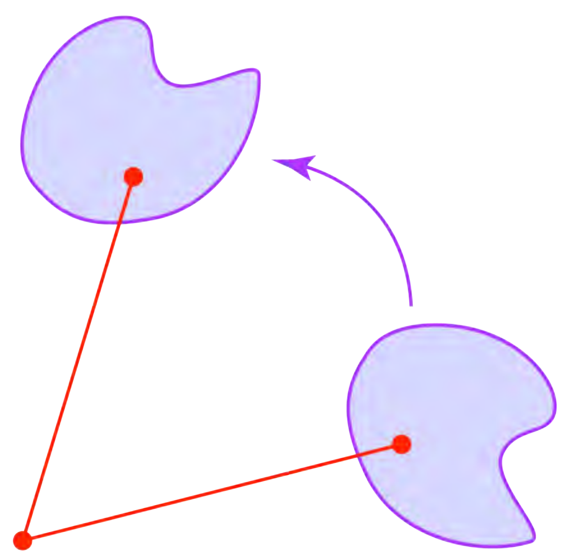


Figure 5.3-1 Each point on the image of a rotated shape is the same distance from the center of rotation as the corresponding point on the preimage.

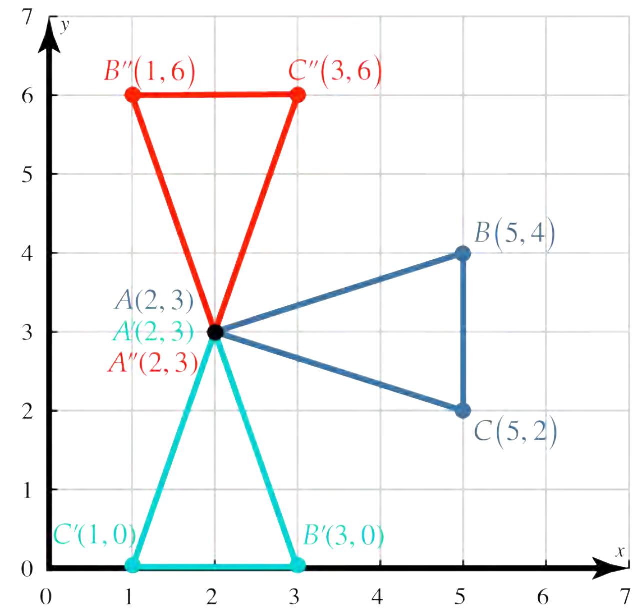
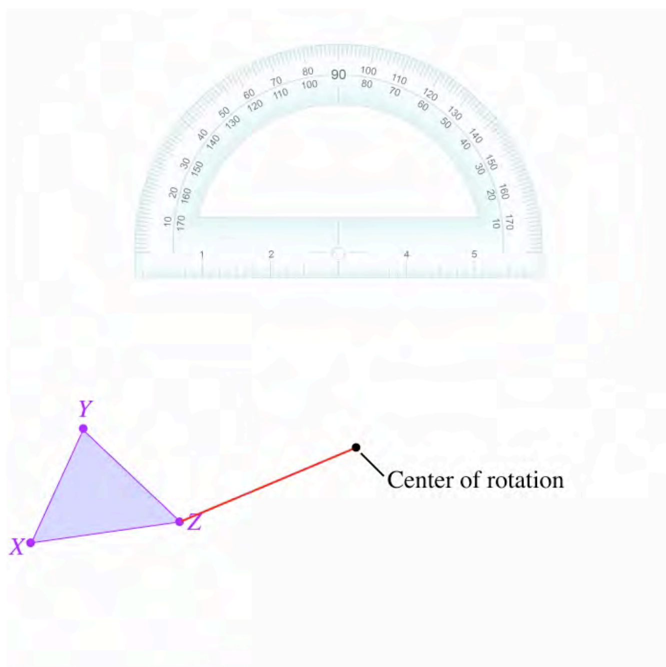


Figure 5.3-2 Triangle *ABC* can be rotated in either direction, but a 90° rotation with no direction specified results in image *A"B"C"*.

Rotating Polygons Using a Protractor

A polygon can be rotated through any angle with a ruler and protractor. First, center the protractor on the center of rotation and line up one vertex of the figure with the 0° line. Second, place a mark at the desired angle of rotation. Third, use the ruler to measure the distance between the center of rotation and the vertex of the preimage. Finally, align the ruler between the center of rotation and the mark made with the protractor, and mark the image vertex the measured distance from the center of rotation. **Animation 5.3-1** illustrates how these steps are used to locate the rotation of a point.

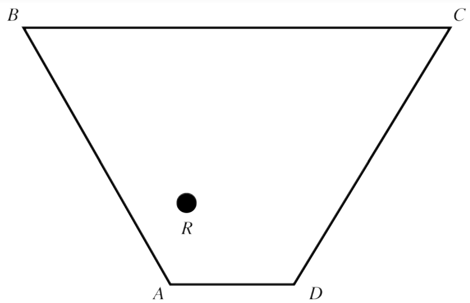
Jump to Degrees



Animation 5.3-1 A ruler and protractor are used to rotate one vertex of a triangle through a 120° angle.

Example problem

Rotate quadrilateral *ABCD* 60° around point *R*.



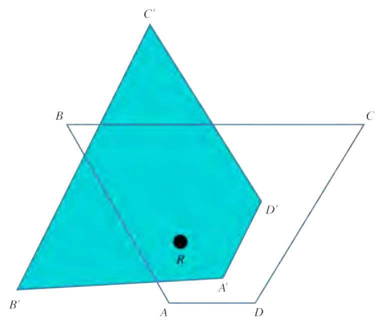
Analyze

The problem gives a diagram of a quadrilateral and asks for a diagram of the image produced when the quadrilateral is rotated around point *R*.

Formulate

Center a protractor on point *R*, and mark a point 60° counterclockwise from point *A*. Then, use a ruler to measure the distance from the center of rotation to that vertex, and copy that distance toward the point marked with the protractor. Repeat the process for the other three vertices.

Determine



Justify

Each vertex of the image was found by measuring the correct angle of rotation with a protractor and measuring the correct distance from the center of rotation with a ruler.

Evaluate

Using the protractor and ruler provided an accurate way to locate the image position of each vertex. The answer is reasonable because each point of the image is the same distance from the center of rotation as the corresponding point of the preimage, and each point has undergone a 60° counterclockwise rotation.

Rotation in the Coordinate Plane

It is possible to produce certain rotations using the coordinates of the vertices of the preimage. The most straightforward of these rotations have rotation angles of 90°, 180°, or 270°. Because each vertex of the preimage rotates in a predictable way around the center of rotation, the corresponding image positions can be calculated.

Point A in **Figure 5.3-3** is two units to the right of point R, which is the center of rotation. Suppose that point A needs to be rotated by 90° around point R. The image A' will still be two units from R, but it will be above R instead of to the right. Rotating 90° again produces point A'', which is to the left of R, and a third 90° rotation produces A''', which is below R.

Suppose that triangle ABC, shown in **Figure 5.3-4**, needs to be rotated 90° around point R. Point A, which is two units to the right of the center of rotation, will produce an image two units above the center, at $(2, 5)$. Point B is four units to the right of the center and one unit above it. Following the counterclockwise rotation for both horizontal and vertical differences produces an image four units above the center of rotation and one unit to its left, at $(1, 7)$. Similarly, since point C is four units to the right and one unit below the center of rotation, its image will be four units above it and one unit to the right, at $(3, 7)$.

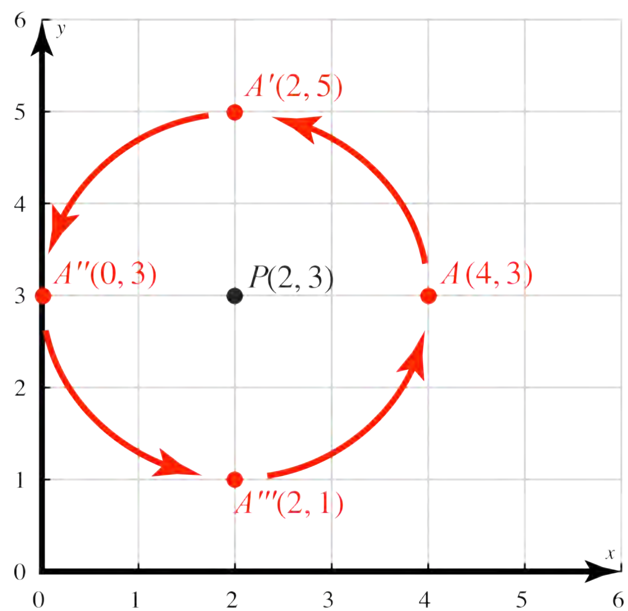


Figure 5.3-3 A series of 90° rotations causes coordinates to the right of the center of rotation to go above it, then to the left, then below, then back to the right.

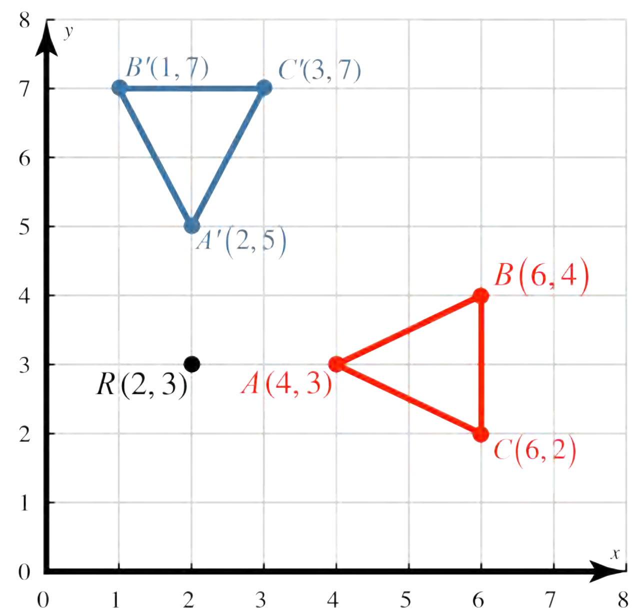


Figure 5.3-4 The coordinates of the vertices of the image triangle can be found from the coordinates of the preimage and the center of rotation.

Calculating Coordinate Positions of Points

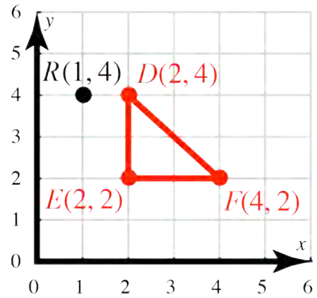
Formulas can be derived for calculating the coordinate positions of points that have been rotated through 90°, 180°, or 270° angles. The x-coordinate of the image is the x-coordinate of the center of rotation (x_R) plus the difference between the y-coordinates of the preimage (y) and the center of rotation $\left(y_R\right)$. A similar formula describes the y-coordinate of the image. The result is the function $(x, y) \rightarrow \left(x_R + y_R - y, y_R + x - x_R\right)$, which describes a 90° rotation where the center of rotation is at $\left(x_R, y_R\right)$. A 180° rotation is described by the function $(x, y) \rightarrow \left(2x_R - x, 2y_R - y\right)$, and a 270° rotation is described by the function $(x, y) \rightarrow \left(x_R + y - y_R, y_R + x_R - x\right)$. These formulas are shown in **Table 5.3-1**.

Rotation	Function
90°	$(x, y) \rightarrow (x_R + y_R - y, y_R + x - x_R)$
180°	$(x, y) \rightarrow (2x_R - x, 2y_R - y)$
270°	$(x, y) \rightarrow (x_R + y - y_R, y_R + x_R - x)$

Table 5.3-1 The formulas for 90°, 180°, and 270° rotations are shown in coordinate function notation.

Example problem

Determine the coordinates of the images of triangle *DEF* after it has been rotated around point R by 90°, 180°, and 270°.



Analyze

The problem gives the coordinates of each vertex of a triangle as well as the center of rotation. It asks for the coordinates of each vertex when the triangle undergoes rotations of 90°, 180°, and 270°.

Formulate

For each rotation, use the corresponding rotation formula for each vertex in order to calculate the x- and y-coordinates of the corresponding vertex of the image.

Determine

First, determine the coordinates of the image rotated 90°.

$(x, y) \rightarrow (x_R + y_R - y, y_R + x - x_R)$	Write the 90° rotation formula.
$(2,4) \rightarrow (1+4-4, 4+2-1)$	Plug in the coordinates for *D*.
$(2,4) \rightarrow (1,5)$	Simplify.
$(2,2) \rightarrow (1+4-2, 4+2-1)$	Plug in the coordinates for *E*.
$(2,2) \rightarrow (3,5)$	Simplify.
$(4,2) \rightarrow (1+4-2, 4+4-1)$	Plug in the coordinates for *F*.
$(4,2) \rightarrow (3,7)$	Simplify.

Next, determine the coordinates of the image rotated 180°.

$(x, y) \rightarrow (2x_R - x, 2y_R - y)$	Write the 180° rotation formula.
$(2,4) \rightarrow (2(1) - 2, 2(4) - 4)$	Plug in the coordinates for *D*.
$(2,4) \rightarrow (0,4)$	Simplify.

Problem continues

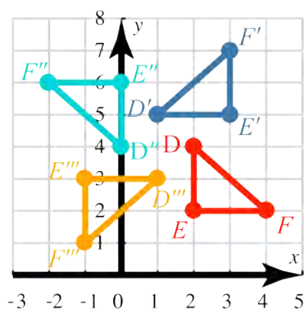
$(2,2) \to (2(1)-2, 2(4)-2)$	Plug in the coordinates for E.
$(2,2) \to (0,6)$	Simplify.
$(4,2) \to (2(1)-4, 2(4)-2)$	Plug in the coordinates for F.
$(4,2) \to (-2,6)$	Simplify.

Finally, determine the coordinates of the image rotated 270°.

$(x,y) \to \left(x_R + y - y_R, y_R + x_R - x\right)$	Write the 270° rotation formula.
$(2,4) \to (1+4-4, 4+1-2)$	Plug in the coordinates for D.
$(2,4) \to (1,3)$	Simplify.
$(2,2) \to (1+2-4, 4+1-2)$	Plug in the coordinates for E.
$(2,2) \to (-1,3)$	Simplify.
$(4,2) \to (1+2-4, 4+1-4)$	Plug in the coordinates for F.
$(4,2) \to (-1,1)$	Simplify.

Justify

When the rotation formulas were followed, the images formed do in fact appear as rotations of the preimage when plotted on a coordinate plane.

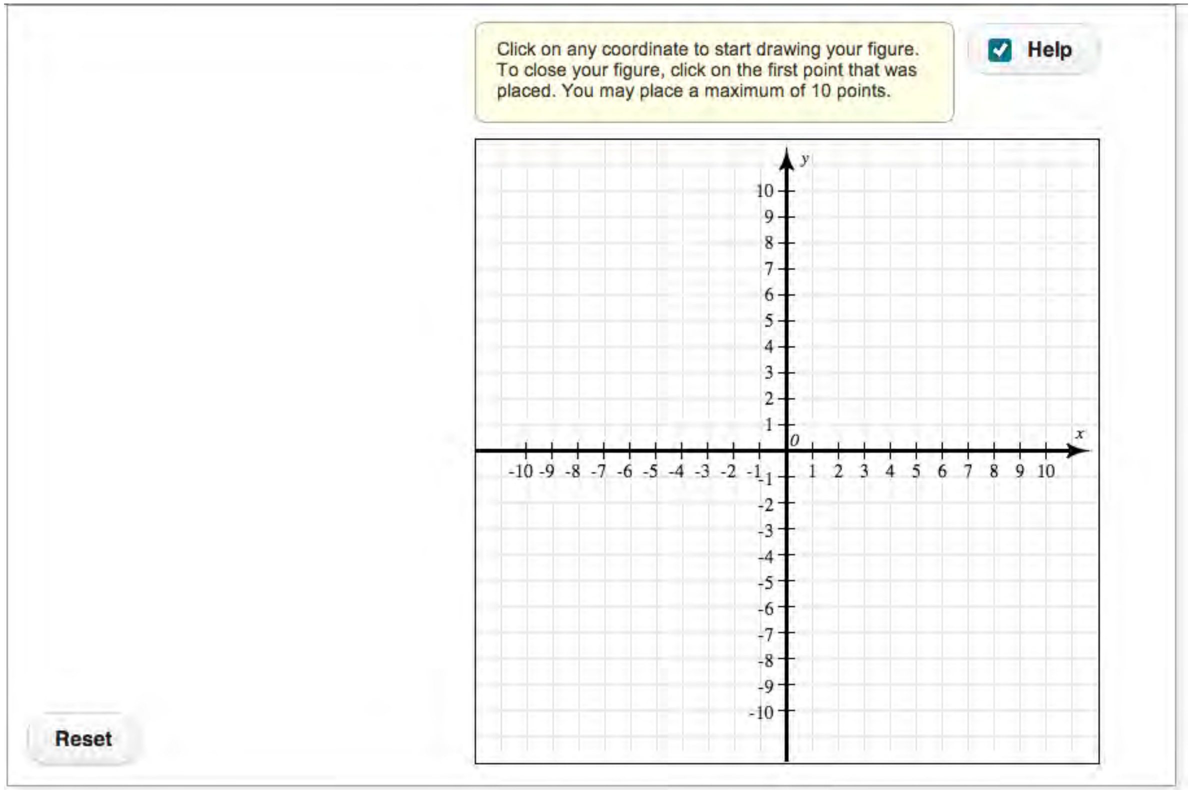


Evaluate

The formulas provide a straightforward way to determine the coordinates of each image vertex. The answer is reasonable because each image appears to be a rotation of the preimage.

Virtual Manipulative - Image Rotation

The most straightforward rotations of geometric figures have rotation angles of 90°, 180°, or 270°. In this virtual manipulative, you will be able to draw your own geometric figure and rotate it across a coordinate plane in both clockwise and counterclockwise directions in 90º increments. All rotations will use the origin as the center of rotation.



You should have been able to rotate any drawn figure across the coordinate plan in 90º increments in both directions. You should have also noticed how the rotations followed a circular path. It would make sense, then, that a rotation of 360º lands the image right back over the preimage.

Properties of Rotations

When a figure is rotated, how it is positioned changes while its shape and size do not. As a result, the image is congruent to the preimage. Rotation is an example of a rigid transformation because the shape and size of the rotating object do not change. **Figure 5.3-5** shows a rectangle that has been rotated around a point. Notice that each side of rectangle *ABCD* is congruent to the corresponding side of rectangle *A'B'C'D'* and that the angles in the preimage are congruent to the angles in the image.

Sometimes, the center of rotation is at the center of the rotating object. In this case, that object spins around its own center. Consider the example of a spinning top. As the top spins, each point on it remains equally distant from the center of the top. Because of the top's symmetry, its appearance does not change even as it rotates. The earth, shown in **Animation 5.3-2**, is another example of an object whose basic shape does not appear to change as it rotates around its own center.

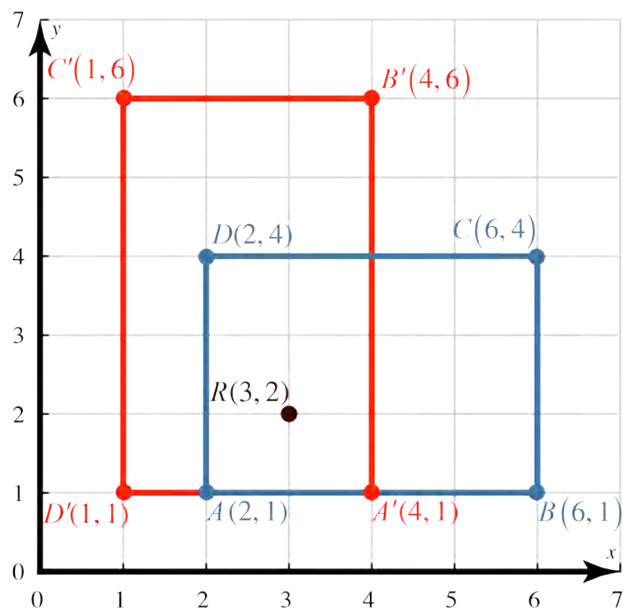
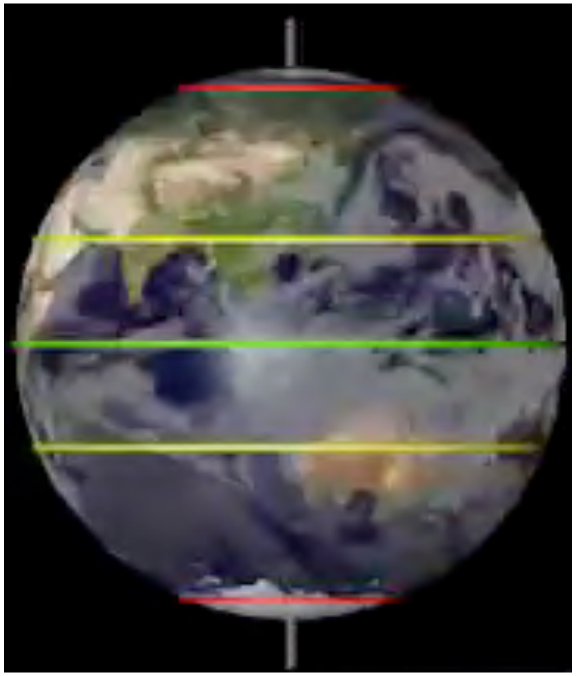


Figure 5.3-5 The image and preimage of a rotation have congruent corresponding parts.



Animation 5.3-2 The earth's shape and size do not change as it rotates around its own center.

Section 5.4

Dilations

Objectives

- Determine the image of a figure in a plane after a dilation, given the center and scale factor
- Use coordinate notation to describe dilations in the coordinate plane with the center of dilation at the origin
- Use the image of a figure after a dilation to determine the preimage

New Vocabulary

- Dilation
- Scale factor
- Center of dilation
- Similarity

A photograph is a common item that is made in different sizes for varying reasons. The same picture can be made into a 4 × 6, 6 × 9, or 8 × 12 in. print. The image is still the same, but everything about it has either gotten larger or smaller. What needs to happen to a 4 × 6 picture to make it into an 8 × 12? What has happened to a 6 × 9 picture to make it a 4 × 6? What operations are being performed?

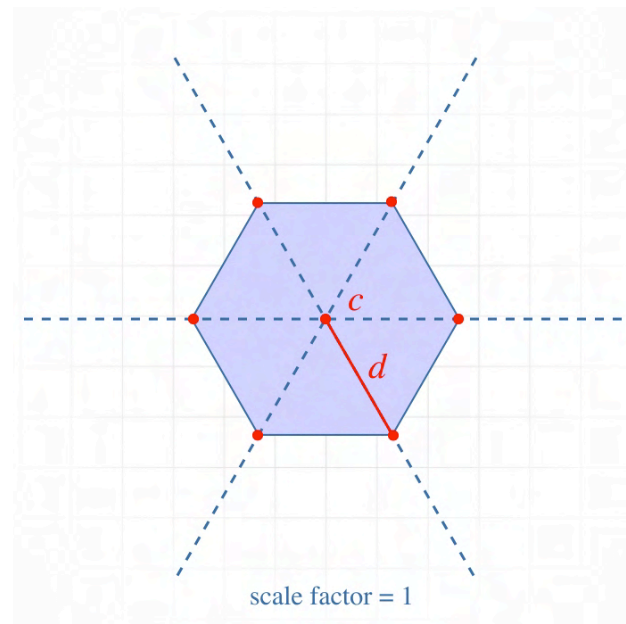
Dilations

A dilation is a uniform enlargement or reduction of a preimage around a given point. The ratio of the image size to the preimage size is known as the scale factor (k). If $k > 1$, then the dilation is an enlargement. If $0 < k < 1$, then the dilation is a reduction. The center of dilation is a static point about which all the other points in the figure are enlarged or reduced. Suppose the center of dilation is C. If P is a point that is not located at the center of dilation, then P' lies on $\overrightarrow{CP}$. Its distance from C is k times the distance CP, or $CP' = k(CP)$. This is illustrated in **Figure 5.4-1**.

Figure 5.4-1 If $CP = 3$ and $k = 2$, then $CP' = 6$ because $CP' = k(CP)$.

Animation 5.4-1 shows how a dilation is performed on a preimage to produce an image. Notice that the distances from the center of dilation to the vertices of the preimage are measured and then multiplied by a scale factor. In this case, the scale factor is 2. If d is the distance from the center of dilation to a certain vertex of the preimage, then measuring a distance $2d$ from the center and placing a new vertex at that point creates the corresponding vertex in the image.

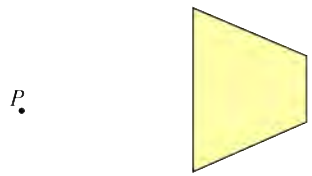
A dilation is a type of uniform scaling, which means that it is a transformation that preserves similarity but not congruence. All corresponding sides of a preimage and image of a dilated figure have the same proportion, and all angles are equal. The only things that change about the figure are its size and position.



Animation 5.4-1 A dilation is created by measuring the distance from the center of dilation to each vertex and then multiplying that distance by the scale factor to find the distance to the vertices of the image.

Example problem

Draw the image of the figure after a dilation about point P by a scale factor of 0.5.



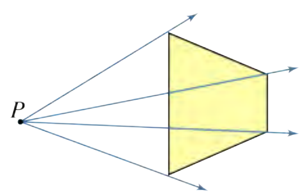
Analyze

The problem presents a polygon and asks for its image formed by a dilation. The scale factor is 0.5, and the center of dilation is at point P.

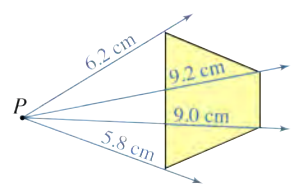
Formulate

Draw a ray from P toward each vertex of the figure. Then, use a ruler to measure the distance to each vertex from point P. Finally, locate the new vertices by multiplying 0.5 by each of these distances and marking the transformed vertex that distance from P on the corresponding ray.

Determine

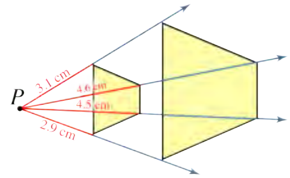


Draw a ray from P toward each vertex.



Measure the distance from P to each vertex.

 Problem continues



Multiply each distance by 0.5, and draw the image vertex that distance from P along the same ray as the corresponding preimage vertex.

Justify

The image was determined by calculating the required distance from the center of dilation to each vertex and then by measuring that distance from the center to place the vertex accordingly.

Evaluate

Using a ruler to measure distances from the center provided a way to create an accurate dilation as long as the measurements were taken carefully. The answer can be verified by measuring each distance from point P to each point in the preimage and making sure each of those distances is half of the distance from P to the corresponding point in the image.

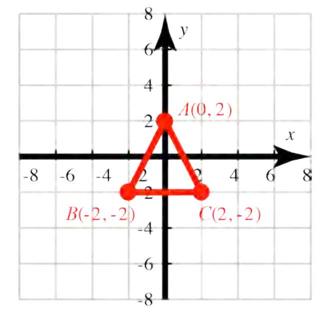
Typically, because the image is not the same size as the preimage, a dilation is not considered an isometry. In order for a dilation to be an isometry, the scale factor must be exactly 1. In this case, each point of the image maps directly onto the corresponding point of the preimage. As a result, the image is not only congruent to the preimage but also is in the same position and orientation.

Dilation in the Coordinate Plane

A dilation in the coordinate plane centered at the origin is found by multiplying the coordinates by the scale factor (k). In coordinate notation, this is represented as $(x, y) \to (kx, ky)$.

Example problem

Graph the image produced if $\triangle ABC$ is dilated about the origin by a scale factor of 3.



Analyze

The problem asks for the image of $\triangle ABC$ after a dilation with a scale factor of 3 centered at the origin.

Formulate

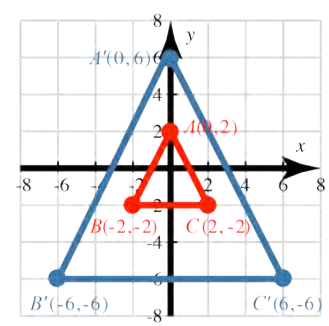
Multiply the x- and y-coordinates of each vertex by 3, and then graph the new figure.

Determine

$A(0, 2) \to A'(0 \cdot 3, 2 \cdot 3)$ $A' = (0, 6)$	Multiply each coordinate of A by 3.
$B(-2, -2) \to B'(-2 \cdot 3, -2 \cdot 3)$ $B' = (-6, -6)$	Multiply each coordinate of B by 3.
$C(2, -2) \to C'(2 \cdot 3, -2 \cdot 3)$ $C' = (6, -6)$	Multiply each coordinate of C by 3.

Justify

Because the dilation is centered at the origin, multiplying each coordinate by the scale factor transformed each point so that it was dilated by an amount equal to the scale factor.



Evaluate

Multiplying the coordinates by the scale factor provided an accurate way to determine the exact position of the image. The answer is reasonable because the scale factor greater than 1 produced an image larger than the preimage.

Centers of Dilations Other Than the Origin

If the center of dilation is not at the origin, then the image cannot be found simply by multiplying each coordinate by the scale factor. In these cases, the displacement from a point in the preimage from the center of dilation must be found first. Then the displacement is multiplied by the scale factor, and this value is added to the center to find the location of the point in the image. Repeating this process for each vertex produces the image of the dilation. This process is described by the transformation $(x, y) \to \left(x_c + n(x - x_c), y_c + n\left(y - y_c\right)\right)$, where the terms with the subscript c refer to the center of dilation, and n refers to the scale factor.

Jump to Translation

Suppose a figure is dilated about the point (–2, 3) by a scale factor of 2. If (x, y) is a point on the figure that needs to be dilated under these conditions, first the displacement between (–2, 3) and (x, y) needs to be found. The horizontal displacement is $x + 2$, and the vertical displacement is $y - 3$. These displacements can now be multiplied by the scale factor, becoming $2x + 4$ and $2y - 6$. Next, these distances need to be added to the coordinates of the center, resulting in the point $(2x + 2, 2y - 3)$.

This process is illustrated in **Figure 5.4-2** with $\triangle ABC$ at points (2, 1), (3, 1), and (2, 3). This triangle is to be dilated about point (1, 1) by a scale factor of 2. The first step is to find the horizontal and vertical displacements to each point from the center of dilation. For the point (2, 3), the displacement is one unit to the right and two units up. The next step is to multiply each displacement by 2. For this point, that produces the displacement of two units to the right and four units up. Finally, add each displacement to the center of dilation to determine the location of each dilated point. For this point, that produces the image point $(3, 5)$. Applying this same process to the other point in the triangle results in the dilated triangle with vertices at (3, 1), (5, 1), and (3, 5).

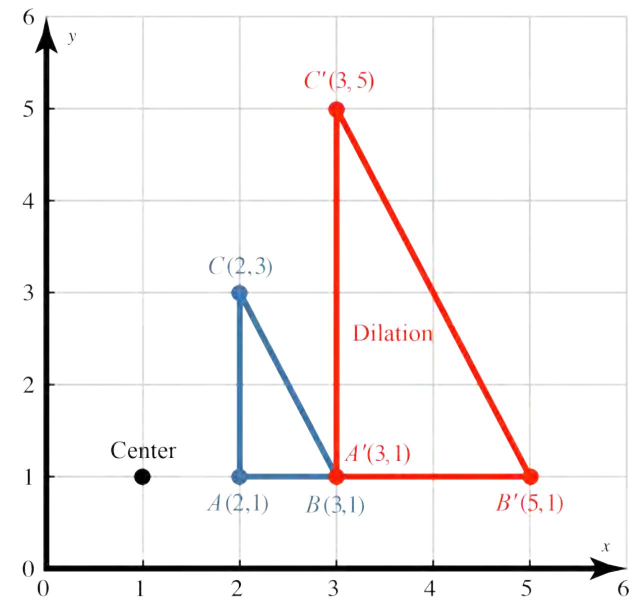
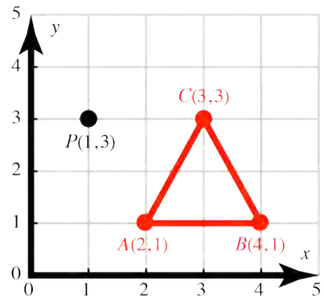


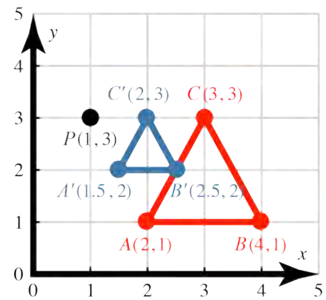
Figure 5.4-2 A dilation centered around a point that is not the origin can be found by multiplying the scale factor by each displacement from the center.

Example problem	Find the image of $\triangle ABC$ produced by a dilation centered at point $P(1,3)$ and having a  scale factor of 0.5.
Analyze	The problem asks for the image of triangle ABC after a dilation by 0.5 centered about $(1,3)$.
Formulate	First, find the horizontal and vertical displacements to each point from the center of dilation. Next, multiply these displacements by the scale factor. Finally, add each of these displacements to the coordinates of the center of dilation.

Determine

From $P(1,3)$ to $A(2,1)$: right 1, down 2	Find the displacement from center.
From P to A': right 0.5, down 1	Multiply the displacement by the scale factor.
A': $(1.5, 2)$	Add the scaled displacement to the coordinates of the center.
From $P(1,3)$ to $B(4,1)$: right 3, down 2	Find the displacement from center.

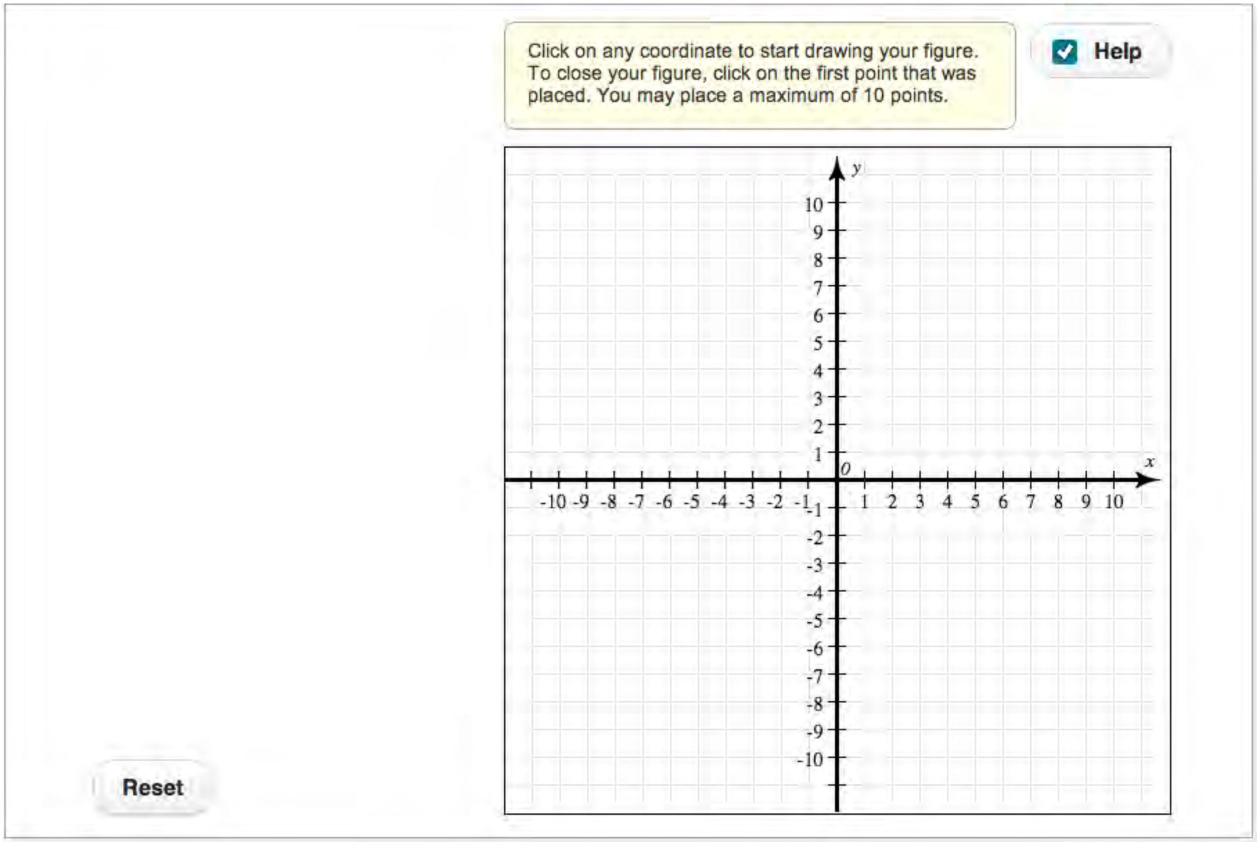
Problem continues

	From P to B': right 1.5, down 1	Multiply the displacement by the scale factor.
	B': $(2.5, 2)$	Add the scaled displacement to the coordinates of the center.
	From $P(1, 3)$ to $C(3, 3)$: right 2, up or down 0	Find the displacement from center.
	From P to C': right 1, up or down 0	Multiply the displacement by the scale factor.
	C': $(2, 3)$	Add the scaled displacement to the coordinates of the center.
Justify		Because the center of dilation is not the origin, the distances from the center of dilation to each vertex of the preimage were found and then doubled.

Evaluate

Dilating one point at a time made the process straightforward. Plotting the image on a graph demonstrated that the image found is a dilation of the preimage.

A dilation in the coordinate plane centered at the origin is found by multiplying the coordinates by the scale factor. In this virtual manipulative, you will draw a geometric figure on a coordinate plane and dilate it about the origin by adjusting the scale factor applied to the coordinates.



In the virtual manipulative, you should have seen how each scale factor changes the coordinates of the image. You should have also noticed that if an entire preimage is in only one quadrant, it remains in that quadrant no matter the scale factor applied to it.

Properties of Dilations

All transformations examined previously were rigid, meaning that they preserved both the shape and size of the figure. Translations, reflections, and rotations are all examples of rigid transformations. Dilations do not fall into this category. Although the orientation of a dilated figure remains the same, the size of the image compared to the preimage is often either smaller (for scale factors less than 1) or larger (for scale factors greater than 1). A nonrigid transformation, therefore, can maintain the shape of the figure while not maintaining its size.

Two triangles are shown in **Figure 5.4-3**. The triangle on the right is produced by a dilation of the triangle on the left. Notice that each angle in the image is congruent to the corresponding angle in the preimage. Also, each side of the second triangle is exactly half the length of the corresponding side of the first triangle. Although the preimage and the image are not congruent, they are similar. Similarity is the property of figures having the same shape but not necessarily the same size. Corresponding angles in similar figures are always congruent, and all pairs of corresponding sides share the same proportion. Because the triangles in **Figure 5.4-3** have congruent angles, and because each side of the second triangle is exactly half the length of the corresponding side of the first triangle, these two triangles are similar. As a result, it can be seen that an image produced by a dilation is similar to its preimage.

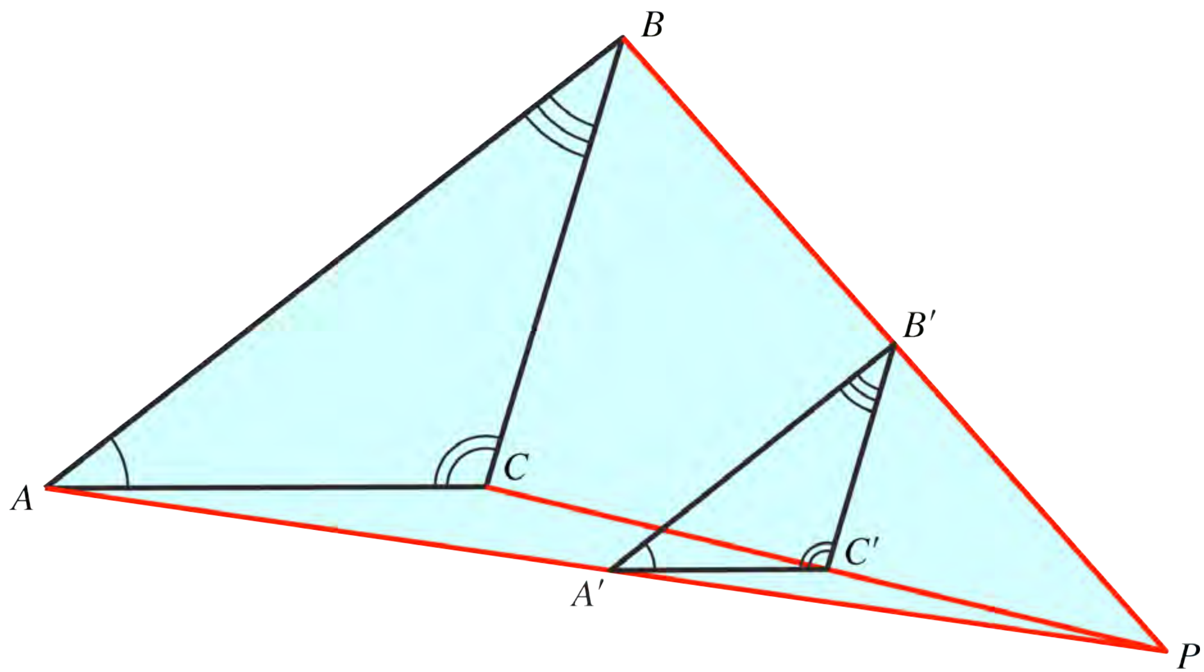


Figure 5.4-3 An image produced by a dilation is similar to its preimage.

Section 5.5

Compositions

Objectives

- Identify the preimage or image of a composition of transformations
- Determine a possible composition of transformations for a given preimage and image
- Describe translations and rotations in terms of reflections

New Vocabulary

- Glide reflection
- Composition (transformation)
- Isometry

What happens if you shift an image, and then shift it again? What if you reflect the image and then shift it? Or translate it and then rotate it? Many transformations can be described in terms of other transformations. Look at the footprints in the sand. No single transform can take one footprint and map it to the next. What two transforms would it take?

Composition of Transformations

The lock in **Figure 5.5-1** is shown in two different positions. In order to get it from the first position to the second, two transformations are necessary. The process of flipping the handle from the top of the lock to the bottom produces a rotation. However, the rotation needs to be followed by a translation in order to produce the second image.

A glide reflection is a transformation that involves both a reflection across a line and a translation parallel to that line. Footprints are an example of a glide transformation. A glide reflection is one type of composition. A composition is a series of two or more transformations that are used together to form an image. A composition could involve any combination of translations, rotations, reflections, dilations, or other transformations.

Jump to Reflection Across a Line

Jump to Translation

Suppose that triangle ABC undergoes a glide reflection that includes a reflection across the y-axis followed by a translation of four units. The first step is to find the image $A'B'C'$ produced by just the reflection. This is done by applying the transformation $(x, y) \rightarrow (-x, y)$. The next step is to perform the translation on $A'B'C'$ in order to obtain the final image ($A''B''C''$). In this example, that translation involves shifting each point four units upward. This is done by applying the transformation $(x, y) \rightarrow (x, y+4)$ to the image of the first transformation. The complete example is shown in **Figure 5.5-2**.

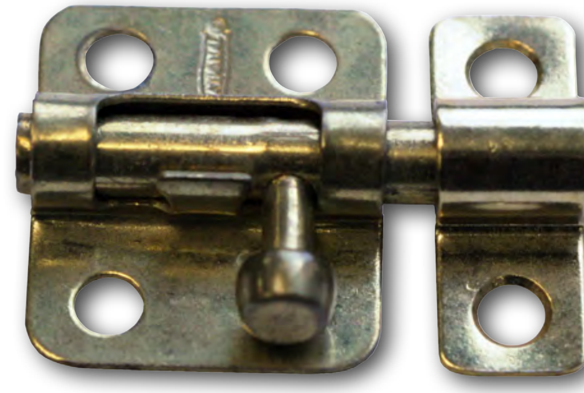


Figure 5.5-1 The handle of this lock undergoes both a rotation and a translation as it is moved from the first position to the second.

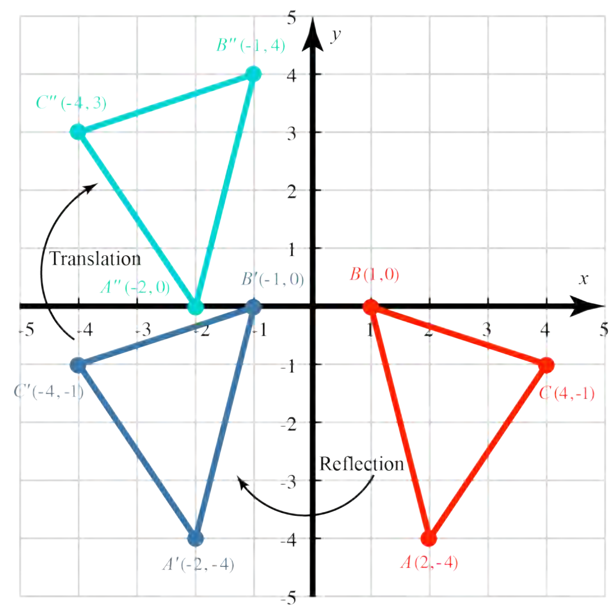
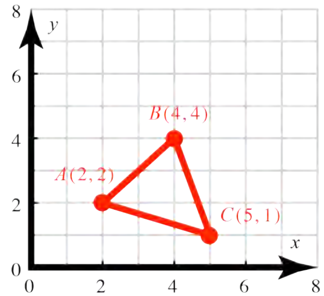


Figure 5.5-2 The glide reflection is performed in two steps: first a reflection and then a translation.

Example problem

Find the vertices of the image of triangle ABC after it undergoes a dilation about the origin by a scale factor of 2 and then a dilation about the point $(6,8)$ by a scale factor of 0.5.



Analyze

The problem gives a triangle and describes two dilations. It asks for the image formed by the composition of dilations.

Formulate

Use the formula for dilation about the origin, $(x,y) \to (nx,ny)$, to locate the vertices of the first dilation. Then, use the general dilation formula, $(x,y) \to \left(x_c + n(x - x_c), y_c + n\left(y - y_c\right)\right)$, to locate the vertices of the second dilation.

Determine

$(2,2) \to \left(2(2), 2(2)\right)$ — Dilate A about the origin to locate A'.

$(2,2) \to (4,4)$

$(4,4) \to \left(2(4), 2(4)\right)$ — Dilate B about the origin to locate B'.

$(4,4) \to (8,8)$

$$(5,1) \rightarrow \left(2(5), 2(1)\right)$$

$$(5,1) \rightarrow (10,2)$$

Dilate C about the origin to locate C'.

$$(4,4) \rightarrow \left(6+0.5(4-6), 8+\frac{1}{2}(4-8)\right)$$

$$(4,4) \rightarrow (5,6)$$

Dilate A' about $(6,8)$ to locate A''.

$$(8,8) \rightarrow \left(6+0.5(8-6), 8+\frac{1}{2}(8-8)\right)$$

$$(8,8) \rightarrow (7,8)$$

Dilate B' about $(6,8)$ to locate B''.

$$(10,2) \rightarrow \left(6+0.5(10-6), 8+\frac{1}{2}(2-8)\right)$$

$$(10,2) \rightarrow (8,5)$$

Dilate C' about $(6,8)$ to locate C''.

Justify

The composition was carried out by performing the two given dilations in the specified order. The image $A''(5,6)$, $B''(7,8)$, $C''(8,5)$ is the result.

Evaluate

The dilation formulas provided a straightforward way to perform the two dilations. The answer is reasonable because it was produced by applying the appropriate formulas.

Applying & Reversing Compositions

Compositions can involve many different types of transformations. Suppose that a triangle, $D(1,1)$ $E(2,4)$ $F(4,2)$, is translated two units to the right and three units upward and then dilated around the origin by a scale factor of 2. The final image can be found in two steps. The first step is to find the image produced by the translation, and the second step is to dilate this image in order to find the final image. The function $(x, y) \rightarrow (x+2, y+3)$ can be used to perform the translation. Applying this function produces the image $D'(3,4)$, $E'(4,7)$, $F'(6,5)$. This image can then be dilated with the function $(x, y) \rightarrow (2x, 2y)$. The result is the second image $D''(6,8)$, $E''(8,14)$, $F''(12,10)$. These images are shown in **Figure 5.5-3**.

Jump to Dilation

Suppose that it is necessary, given an image, to find the preimage from which it was produced. This can be done by reversing the steps used to produce the image. For example, if an image was produced by applying first a reflection and then a translation, the translation must be undone first and the reflection second. Each individual process must also be reversed. For instance, if forming the image involved translating the preimage by three units to the right, this translation must be undone with a translation of three units to the left.

Suppose it is known that the triangle in **Figure 5.5-4** was obtained by translating a preimage two units to the right and three units upward and then dilating it around the origin by a scale factor of 2. Finding the preimage requires first undoing the dilation with the transformation $(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$ and then undoing the translation with the transformation $(x, y) \rightarrow (x-2, y-3)$.

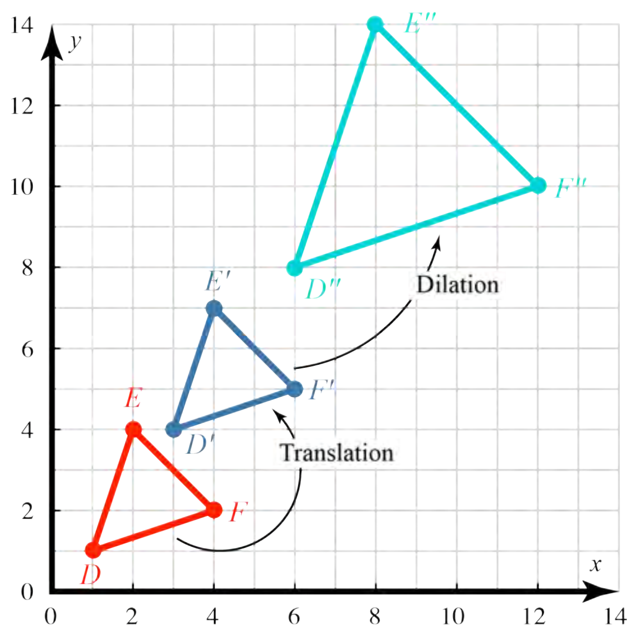


Figure 5.5-3 This figure shows the composition that is performed by applying a translation and then a dilation.

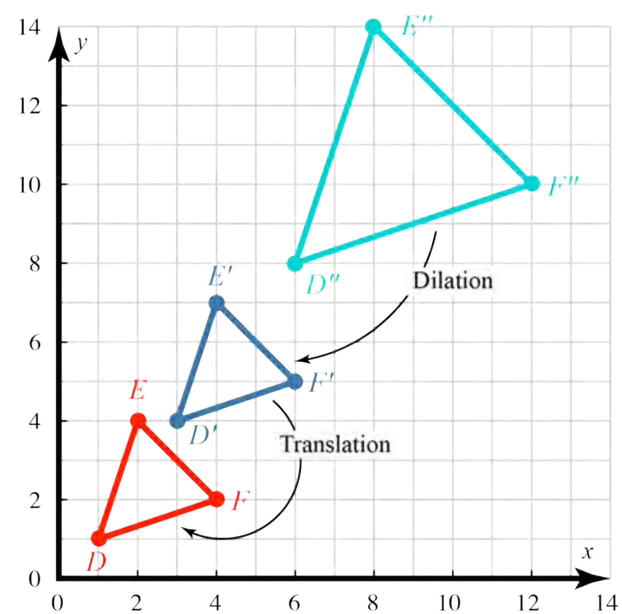
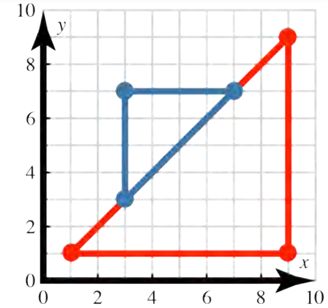


Figure 5.5-4 Locating the preimage requires reversing the steps used to form the image.



Example problem	Describe two different compositions that could be used to transform the smaller triangle into the larger one.
Analyze	The problem shows two triangles and asks for two different compositions of transformations that could be applied to the smaller one in order to transform it into the larger one.
Formulate	First, identify the ways in which the first triangle is unlike the second. Next, identify the transformations that could be used in order to eliminate each of these differences. Apply the transformations in the way that will align the triangles most efficiently.
Determine	The triangles are different in size, position, and orientation. Dilate the smaller triangle by a scale factor of 2 to make the triangles congruent. Choose $(5, 5)$ as the center of dilation in order to align the hypotenuses. Next, rotate the first triangle by 180° in order to give it the same orientation as the second triangle. Use $(5, 5)$ as the center of rotation in order to completely align the triangles. Alternatively, dilate the smaller triangle by a scale factor of 2 with the center of dilation at $(5, 5)$ to make the triangles congruent, and then reflect the triangle across the line $y = x$ in order to align the triangles.
Justify	Many different compositions could be used to align these triangles. The dilation was necessary in order to make the triangles the same size. Because reflecting this particular triangle would not change its original shape, reflecting and rotating it would have the same effect, and either transformation could be used in order to change the orientation. These transformations were performed in such a way as to eliminate the need for a final translation.
Evaluate	Analyzing the differences between the triangles facilitated finding the appropriate transformations. The answers are reasonable because both methods completely align the triangles.

Isometry

Several types of transformations are shown in **Figure 5.5-5**. Notice that the images formed by the translation, rotation, and reflection are all congruent to the preimage. The transformations that produce congruent images are called isometries. An isometry is a transformation in which the distance between any two points on the preimage is equal to the distance between the corresponding points on the image. Every isometry produces an image congruent to the preimage, and every transformation that produces such an image is an isometry. In this sense, an isometry is the same as a rigid transformation. Translations, rotations, and reflections are all isometries, but dilations are not.

Suppose that the preimage in **Figure 5.5-5** undergoes two transformations: first a reflection and then a rotation. The image would be in a different orientation than any of the images shown in the figure, but it would still be congruent to the preimage. Any combination of translations, rotations, and reflections produces an image that is congruent to the preimage. Another way to say this is that any composition of isometries is an isometry. This idea is stated formally in **Theorem 5.5-1** and proven in **Proof 5.5-2**.

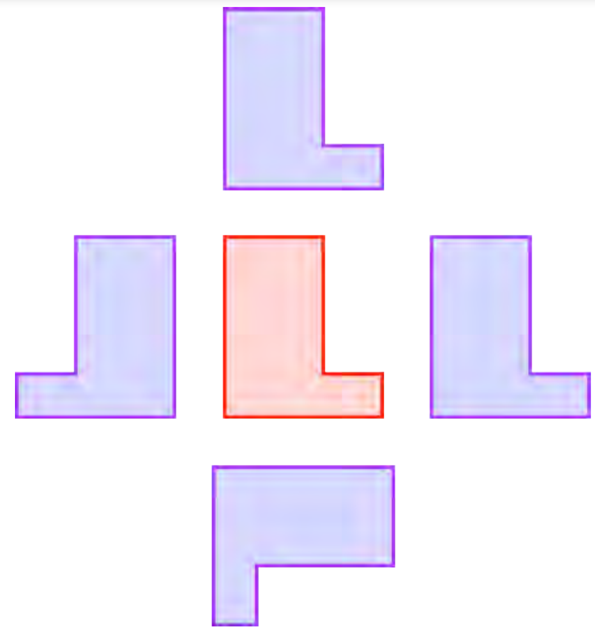


Figure 5.5-5 This figure shows many different transformations. Translations, rotations, and reflections are isometries. Dilations are not isometries.

Proof 5.5-2 Proof of the composition of isometries

Given Preimage A, image A', image A''
$A \rightarrow A'$ is an isometry
$A' \rightarrow A''$ is an isometry
Prove $A \rightarrow A''$ is an isometry

Statements	Reasons
$A \cong A'$	Definition of isometry
$A' \cong A''$	Definition of isometry
$A \cong A''$	Transitive property
$A \rightarrow A''$ is an isometry	Definition of isometry

Theorem 5.5-1 The composition of isometries theorem

If each transformation in a composition of two or more transformations is an isometry, then the composition is also an isometry.

Composition of Reflections

Suppose that a triangle labeled *A*, *B*, and *C* at its vertices is reflected across a line. Although the new image is congruent to the preimage, the image formed by this reflection is flipped. That is, the vertices are labeled counterclockwise instead of clockwise. Now suppose that the shape is reflected again, this time across a second line that is parallel to the first. The image now formed is also congruent with the initial preimage, but the vertices are once again labeled clockwise. This process, both the first and second reflections, is illustrated in **Figure 5.5-6**.

Figure 5.5-6 reveals that not only are the vertices in the second image labeled in the same order as in the preimage, but the figures also have the same orientation. The only difference between the preimage and this second image is the location. As a result, the second image could also be produced by a translation of the preimage. As **Theorem 5.5-3** states, reflecting a figure twice across parallel lines results in a translation of the original figure.

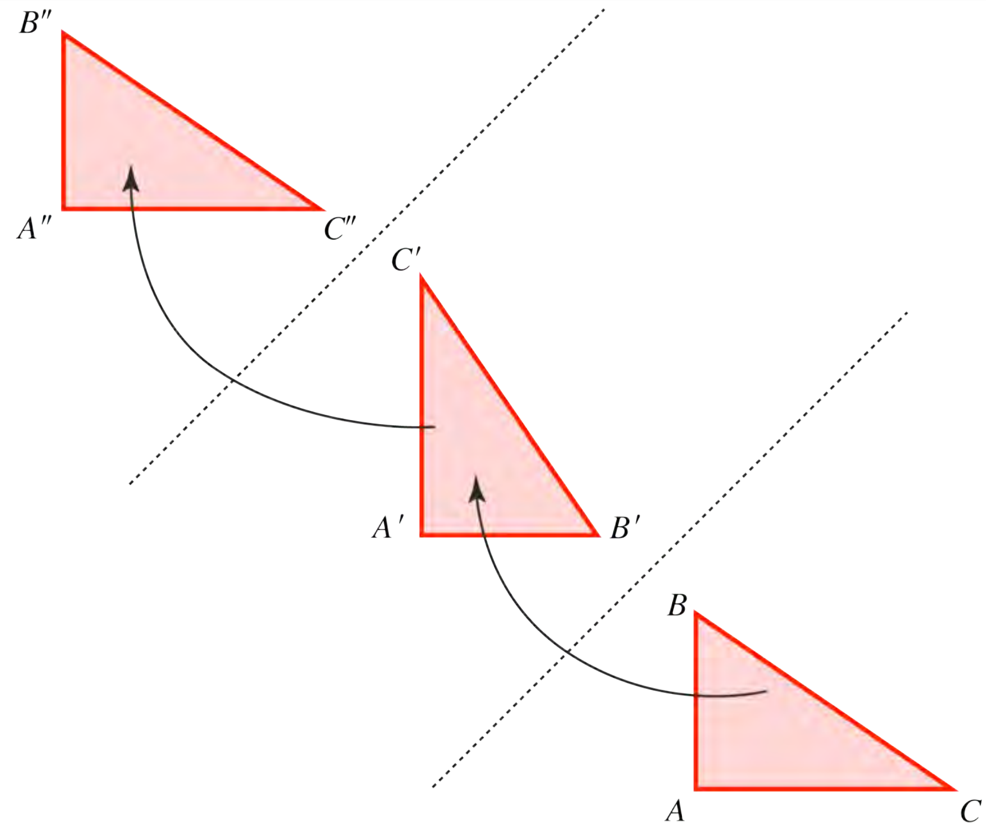


Figure 5.5-6 Reflecting an object twice over parallel lines results in a translation of the preimage.

Theorem 5.5-3 Reflections in parallel lines theorem

A transformation consisting of two reflections across parallel lines is equivalent to a translation. The translation vector is perpendicular to the lines, and its length is 2 times the distance between the lines.

Reflecting an Image Across Two Intersecting Lines

Suppose that a shape such as the one shown in **Figure 5.5-7** is reflected across a line and then across a second line that intersects the first. The result is similar to the image formed by reflections across parallel lines in that the vertices are labeled in the same direction as they are in the preimage. However, the image is in a different orientation than the preimage.

Because the image formed by the two reflections is congruent to the preimage but in a different orientation, this image could be formed by a rotation of the preimage. As **Theorem 5.5-4** states, reflecting a figure twice across intersecting lines results in a rotation of the original figure.

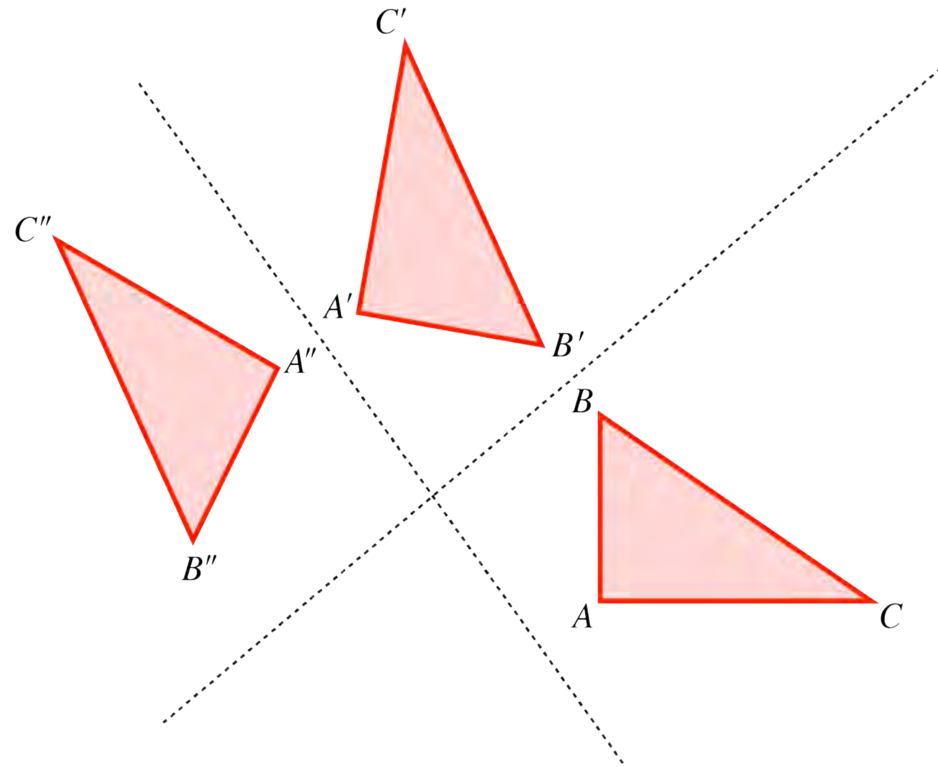


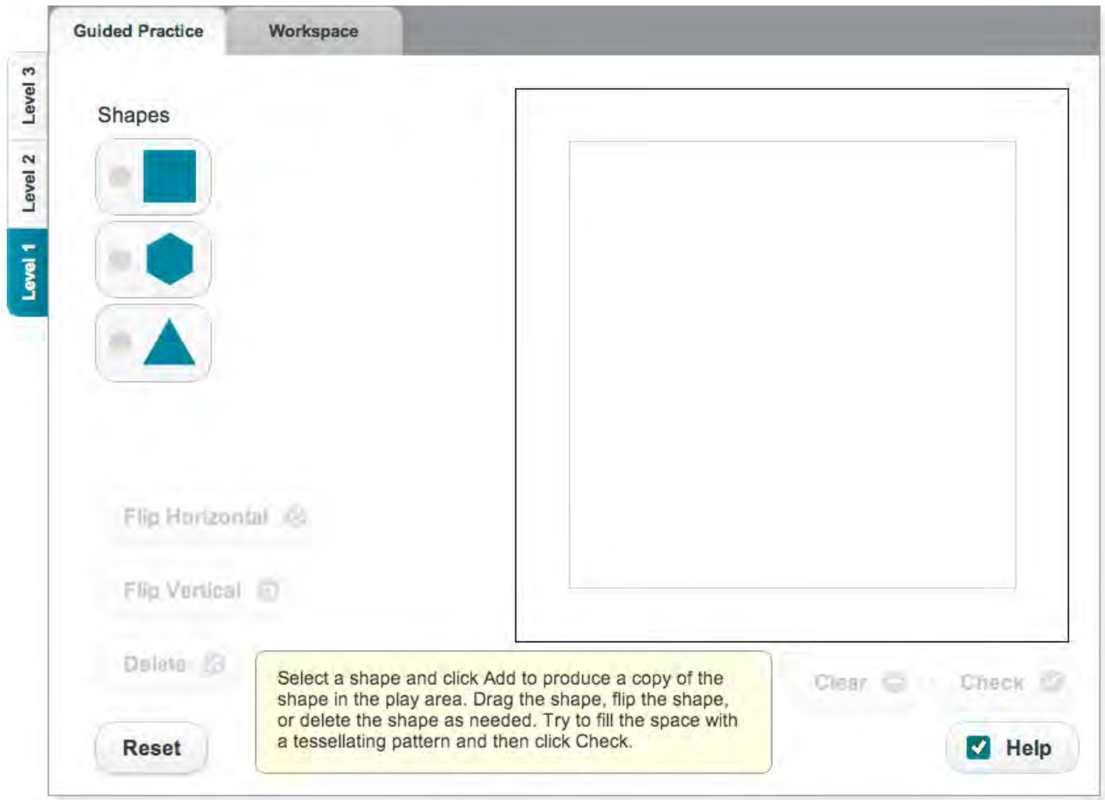
Figure 5.5-7 Reflecting an object twice over intersecting lines results in a rotation of the preimage.

Theorem 5.5-4 Reflections in intersecting lines theorem

A transformation consisting of two reflections across intersecting lines is equivalent to a rotation. The center of rotation is the point of intersection of the lines, and the angle of rotation is 2 times the measure of the acute or right angle formed by the intersecting lines.

Virtual Manipulative - Tessellations

A tessellation is the tiling of a plane using geometric shapes without overlaps or gaps. In this virtual manipulative, you will be able to use a given or created geometric shape to create a tessellating pattern that fills the space using translations, and horizontal and vertical flips.



You should have been able to create a tessellating pattern that filled the space, free of overlaps or gaps, using only translations and flips. You should have also been able to draw your own geometric shape and attempt to tessellate them. Why were some of the shapes you created able to be tessellated, and some not?

Section 5.6

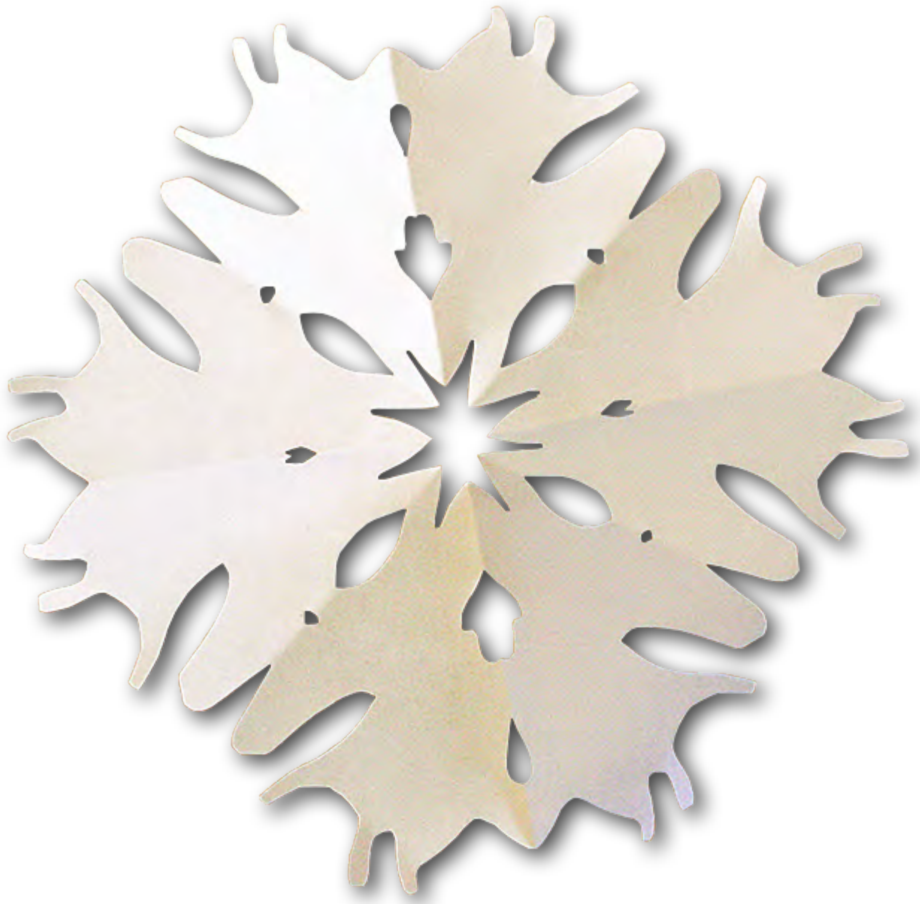
Symmetry

Objectives

- Identify and classify symmetry in two-dimensional and three-dimensional objects

New Vocabulary

- Symmetry
- Line symmetry
- Reflection symmetry
- Line of symmetry
- Axis of symmetry
- Rotational symmetry
- Radial symmetry
- Center of symmetry
- Point of symmetry
- Order (symmetry)
- Magnitude (symmetry)
- Plane symmetry
- Axis symmetry
- Solid of revolution



This paper snowflake was created by folding a piece of paper into eight layers and then making a few cuts. By this process, each section of the snowflake was cut the same way, making a repeating pattern all the way around. If this snowflake were rotated by 180°, would you be able to tell the difference? If it were folded in half, would the two halves perfectly fit together?

Types of Symmetry

A figure has symmetry if there is a rigid transformation that maps the figure back onto itself. That is, a figure has symmetry if, after a rigid transformation, the image and the preimage have the same shape, size, orientation, and position. Many plants and animals exhibit symmetry. The butterfly in **Figure 5.6-1** is an example of this. If this butterfly were reflected across its centerline, the image created would be identical to the original butterfly. The butterfly's symmetry is associated with reflection.

A different type of symmetry is shown in **Figure 5.6-2**. In this example, reflecting the mosaic around its centerline does not produce an image identical to the original mosaic. However, rotating the mosaic by 180° does produce an image that is identical to the original. The mosaic's symmetry is associated with rotation.

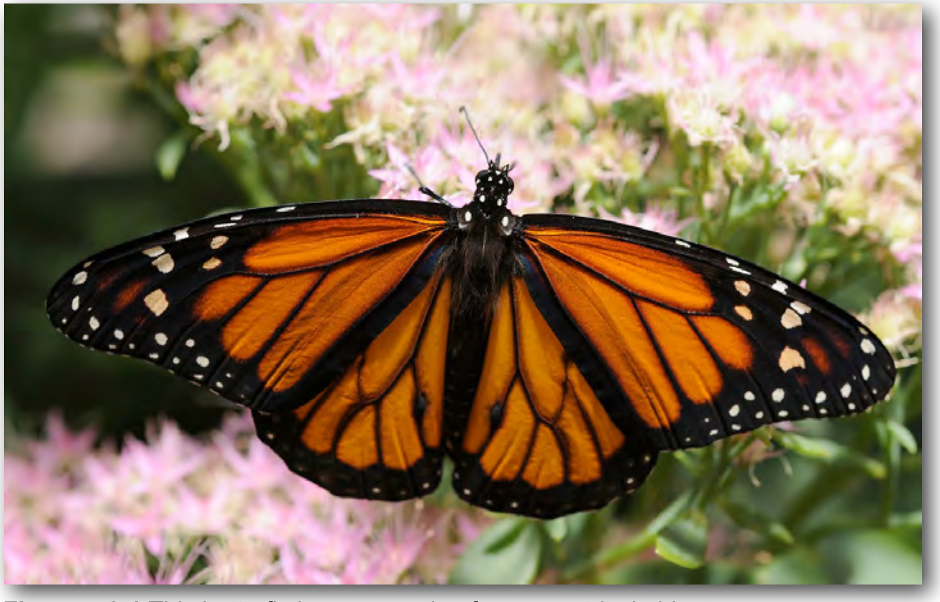


Figure 5.6-1 This butterfly is an example of a symmetrical object.

Figure 5.6-2 A 180° rotation around the center of this mosaic produces a result identical to the original.

Lines of Symmetry

Line symmetry is a type of symmetry in which a reflection of the preimage produces an image with identical position and orientation. Line symmetry is also known as reflection symmetry. The line across which the preimage is reflected is known as the line of symmetry or the axis of symmetry. A drawing of a butterfly is shown in **Figure 5.6-3**. Notice the dotted line running down the center of the butterfly. This butterfly is an example of line symmetry, and the dotted line through its center is the line of symmetry.

Jump to Reflection Across a Line

Suppose that this butterfly were folded at the line of symmetry. The two sides would then completely align, and it would appear as seen in **Figure 5.6-4**. Notice that the central vertical line is the only place where the butterfly could be folded in order to achieve this effect. It would be possible to fold the butterfly across other lines, but in no other case would the two sides of the butterfly completely align. As a result, the butterfly has just a single line of symmetry.

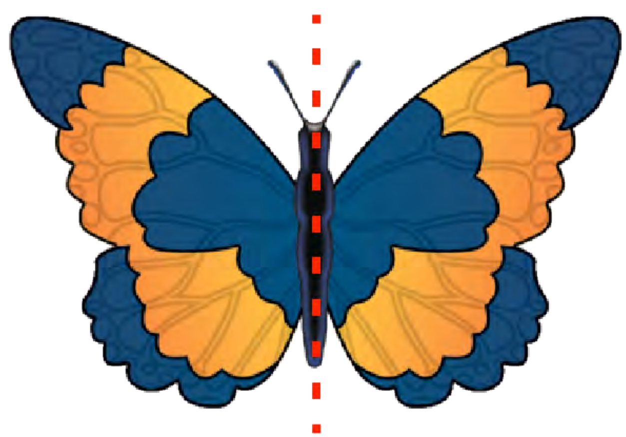


Figure 5.6-3 This drawing of a butterfly demonstrates reflection symmetry with the line of symmetry shown down the center of the drawing.

Figure 5.6-4 When this drawing of a butterfly is folded at the line of symmetry, the two sides completely align.

Multiple Lines of Symmetry

Some shapes have more than one line of symmetry. For these shapes, folding over any of these lines produces two halves that align completely. For example, many types of flowers have multiple lines of symmetry. **Figure 5.6-5** shows an example of such a flower. Notice that a line drawn from the tip of any of the petals through the center of the flower is a line of symmetry. This flower has five such lines, one corresponding to each petal.

Figure 5.6-5 This flower has five different lines of symmetry.

In order to determine the number of lines of symmetry an object has, consider how many different ways it can be folded. The flower in **Figure 5.6-5** can be folded along the centerline of each petal, and each way, the two halves entirely match. As a result, this flower has five lines of symmetry. The number of lines of symmetry can be predicted for certain shapes. For instance, consider the polygons shown in **Figure 5.6-6**. Notice that the pentagon has five lines of symmetry, and the hexagon has six. For any regular polygon, the number of lines of symmetry is equal to the number of sides.

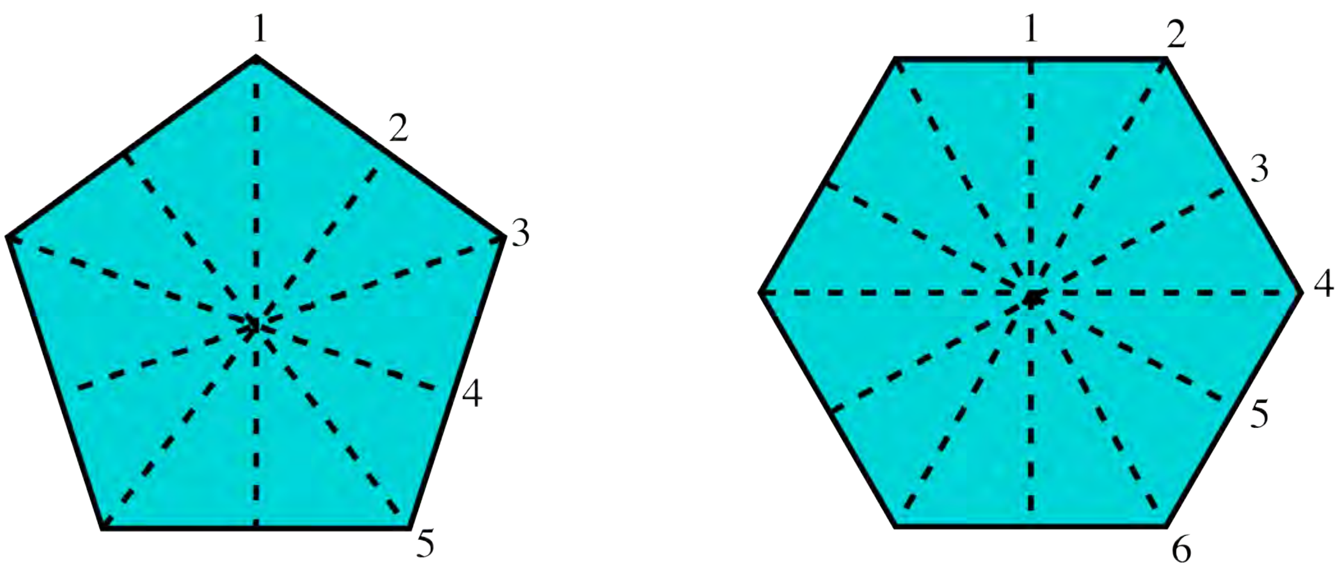


Figure 5.6-6 The number of lines of symmetry is equal to the number of sides on the polygon.

Zero Lines of Symmetry

Many objects, including some that appear to have a form of symmetry, do not have any lines of symmetry. For example, consider a parallelogram that is not equiangular or equilateral. Although several lines could be drawn to divide the parallelogram in half, none of these functions as a line of symmetry. A parallelogram such as this is shown in **Figure 5.6-7**. Notice that when the parallelogram is folded along any of the lines that divide the shape in half, the two sides do not completely align.

As another example, consider the differences that exist with respect to symmetry between a square and a rectangle. A square, which is a regular polygon, has four lines of symmetry. A rectangle, however, has only two lines of symmetry. The diagonals of a rectangle divide the shape in half, but they are not lines of symmetry.

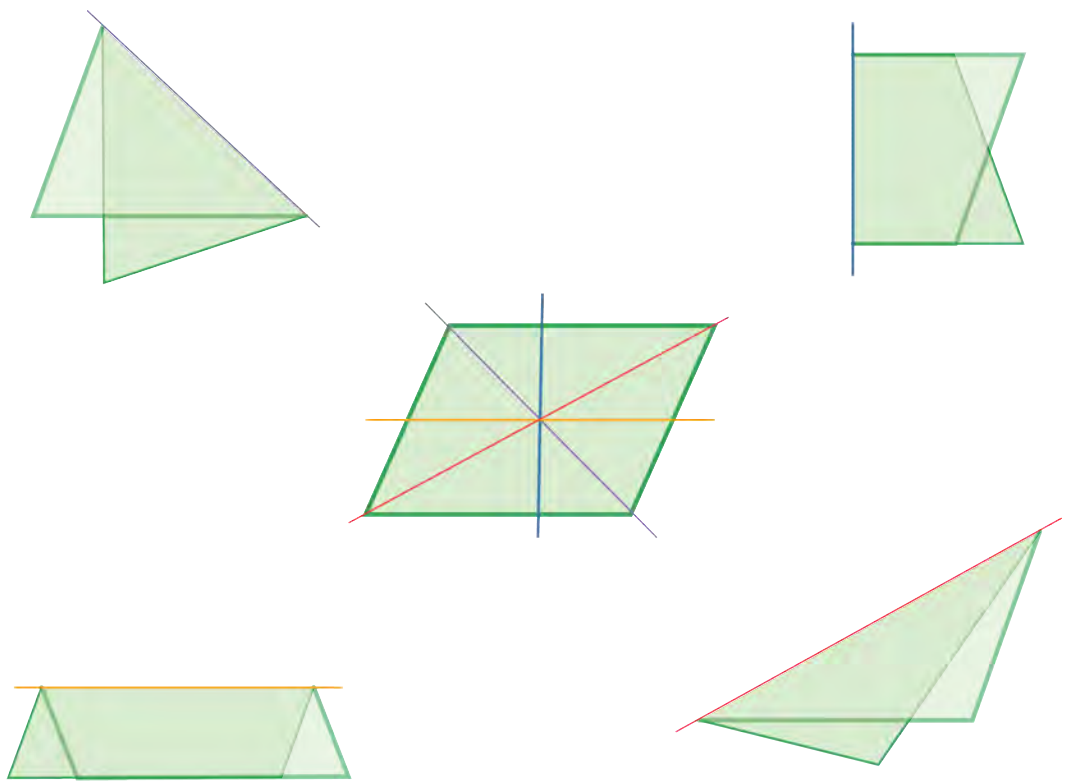


Figure 5.6-7 This parallelogram has no line along which it can be folded in order to make the two sides completely align.

Angle of Rotation

Rotational symmetry is a type of symmetry in which a rotation of the preimage produces a congruent image with the same position and orientation. Rotational symmetry may also be referred to as radial symmetry, for example, in relation to biological organisms. The point around which the preimage is rotated is known as the center of symmetry or the point of symmetry. A picture of a starfish is shown in **Figure 5.6-8**. Notice that there are several angles through which it can be rotated around its center with no change in its appearance.

Figure 5.6-8 This starfish exhibits rotational symmetry.

Jump to Properties of Rotation

The order of symmetry is the number of positions to which a figure can be rotated without any change in its appearance. The magnitude of the symmetry is the measure of the smallest angle through which the figure must be rotated in order to achieve an identical appearance. The starfish's order of symmetry is 5 because it can be in any of five different positions—any of the five points can be at the top—while maintaining the same appearance.

Because there are five angles that contribute to the symmetry of the starfish, each rotation is equal to one-fifth of the total circle: $360°/5 = 72°$. Therefore, the magnitude of this symmetry is 72°. If the legs of the starfish were labeled, then it would be seen that making five 72° rotations returns the starfish to its original position. This pattern is always true. That is, the magnitude of any rotational symmetry can be calculated as 360° divided by the order.

Recall that parallelograms that are not equilateral or equiangular do not exhibit line symmetry. The same parallelogram, however, possesses rotational symmetry. Rotating it by 180° results in an image that cannot be distinguished from the original parallelogram. As a result, the parallelogram, as shown in **Figure 5.6-9**, exhibits rotational symmetry with an order of 2 and magnitude of 180°.

Figure 5.6-9 A parallelogram rotated by 180° around its center appears identical to the original parallelogram.

Rotational Symmetry

The order of a rotationally symmetric shape can be found by examining the figure. For example, the star in **Figure 5.6-10** has 10 points. Any of these 10 points could be positioned at the top of the figure, giving the star 10 identical orientations. This number is the order of the star's symmetry. The order can be found by counting how many times the repeating pattern occurs, spaced regularly around the shape. The magnitude of this symmetry is $\frac{360°}{10}$, which is 36°.

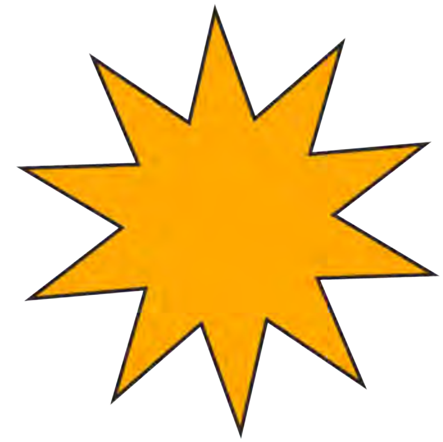


Figure 5.6-10 The order of symmetry for this star is equal to the number of points of the star.

Some shapes that have rotational symmetry also have line symmetry. For instance, a 10-pointed star has both line symmetry and rotational symmetry. An example is shown in **Figure 5.6-11**. Notice that there is a line of symmetry through each pair of opposite points (five lines in all), and a line of symmetry also goes through each of the inner vertices between the points (another five lines). This makes a total of 10 lines of symmetry, which is the same as the order of the rotational symmetry. For any shape that has both line symmetry and rotational symmetry, the number of lines of symmetry is equal to the order of the rotational symmetry.

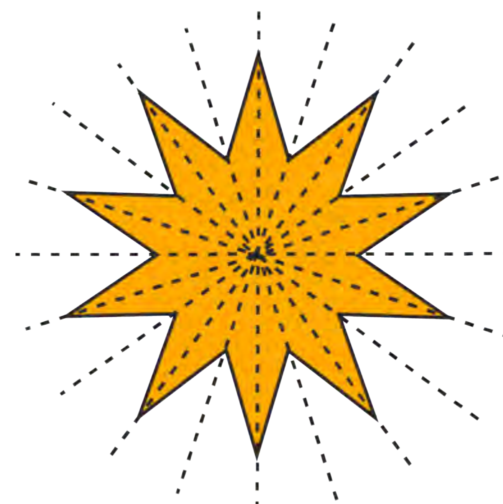


Figure 5.6-11 This star has 10 lines of symmetry, and its rotational symmetry also has an order of 10.

Many shapes do not have rotational symmetry. These include many shapes that do have line symmetry. For example, the trapezoid shown in **Figure 5.6-12** has line symmetry but not rotational symmetry. If the trapezoid is rotated, it must go through an entire 360° rotation before it regains its original appearance.

Figure 5.6-12 The trapezoid does not have rotational symmetry.

Three-Dimensional Symmetry

Three-dimensional objects can also exhibit symmetry. Consider the eyeglasses shown in **Figure 5.6-13**. If a vertical plane were drawn through the center of the bridge, and if the left half of the pair of glasses were reflected across this plane, then it would fall exactly on top of the right half of the glasses.

Figure 5.6-13 The right half of this pair of eyeglasses is a reflection of the left half.

Plane symmetry is a type of symmetry in which a three-dimensional object can be divided by a plane into two halves that are mirror images of each other. The eyeglasses in **Figure 5.6-13** are an example of plane symmetry because the left half is a mirror image of the right half.

Axis symmetry is a type of symmetry in which a three-dimensional object can be rotated around an internal axis in order to achieve an appearance identical to its original appearance. The airplane propeller shown in **Figure 5.6-14** is an example of axis symmetry. If the propeller were rotated by any multiple of 120°, the result would be identical to the original state. However, the propeller does not have plane symmetry. Because of the three-dimensional curvature of the blades, there is no plane that can divide the propeller into two mirror-image halves.

Figure 5.6-14 The propeller exhibits axis symmetry but not plane symmetry.

Plane symmetry is the three-dimensional version of line symmetry. Both are produced by reflection. Just as one-half of a two-dimensional object with line symmetry can be considered a reflection of the other half, in the same way, one-half of a three-dimensional object with plane symmetry can be considered a reflection of the other half. Similarly, axis symmetry is the three-dimensional version of rotational symmetry. Both are produced by rotation. In two-dimensional rotational symmetry, a figure is rotated around a central point to a position in which it appears identical. In axis symmetry, a three-dimensional figure is rotated around a central line to a position in which it appears identical.

Symmetry in Real Objects

Many real three-dimensional objects are symmetrical. For example, even though it may not be apparent at first, the hammer shown in **Figure 5.6-15** has plane symmetry. To visualize the symmetry, imagine viewing the hammer from head on as if looking at its striking surface. Other examples of plane symmetry include an hourglass and a carton of eggs.

There are also many real objects that have axis symmetry. Many patterned bowls, such as the one shown in **Figure 5.6-16**, have axis symmetry. Other examples of axis symmetry include playing cards and pinwheels.

Figure 5.6-15 This hammer has plane symmetry. To visualize the symmetry, imagine viewing the hammer from head on.

Figure 5.6-16 This bowl has axis symmetry.

Solid of Revolution

Suppose that a triangle such as the one shown in **Figure 5.6-17** is rotated around an axis. This motion creates a three-dimensional cone. A three-dimensional solid created by rotating a plane figure around an axis is called a solid of revolution. If the solid were to be cut down the axis of symmetry in order to view the cross section, that cross section would be the rotated plane figure (the red triangle in **Figure 5.6-17**) along with its reflection across the axis of rotation. Consequently, the plane figure that is rotated is half of the cross section of the solid.

The three-dimensional shape of many real objects can be matched by the solid of revolution of a two-dimensional shape. Indeed, some objects are actually created by rotating something around an axis. One example is pottery made on a pottery wheel, as shown in **Figure 5.6-18**. The pot is made by spinning the clay around a central axis. Anything the potter does to shape the clay takes effect all the way around the circumference of the pot. As a result, the pot obtains a form of axis symmetry. The object appears identical when rotated to any angle.

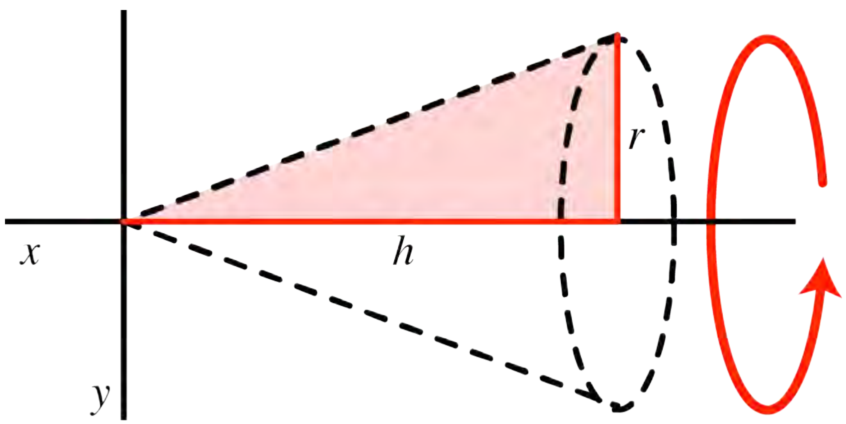


Figure 5.6-17 Rotating this triangle around the *x*-axis creates a three-dimensional cone.

Figure 5.6-18 Pottery is made on a pottery wheel by revolving the clay around a central axis.

Real-World Solids of Revolution

Wooden and metal solids of revolution can be made on a lathe, such as the woodworking lathe shown in **Figure 5.6-19**. The lathe spins the piece of wood or metal, which can be shaped as it spins. This process produces an object with perfect radial symmetry. Baseball bats, table legs, and wooden bowls can all be made by this process.

There are many examples of objects whose shapes are solids of revolution even though they are not actually produced by anything revolving. A flying disk, as shown in **Figure 5.6-20**, has this type of shape. Neglecting any writing or markings on its surface, its appearance is unchanged by a rotation of any angle. Other examples of such objects are Hula-Hoops, plumbing pipes, and CDs.

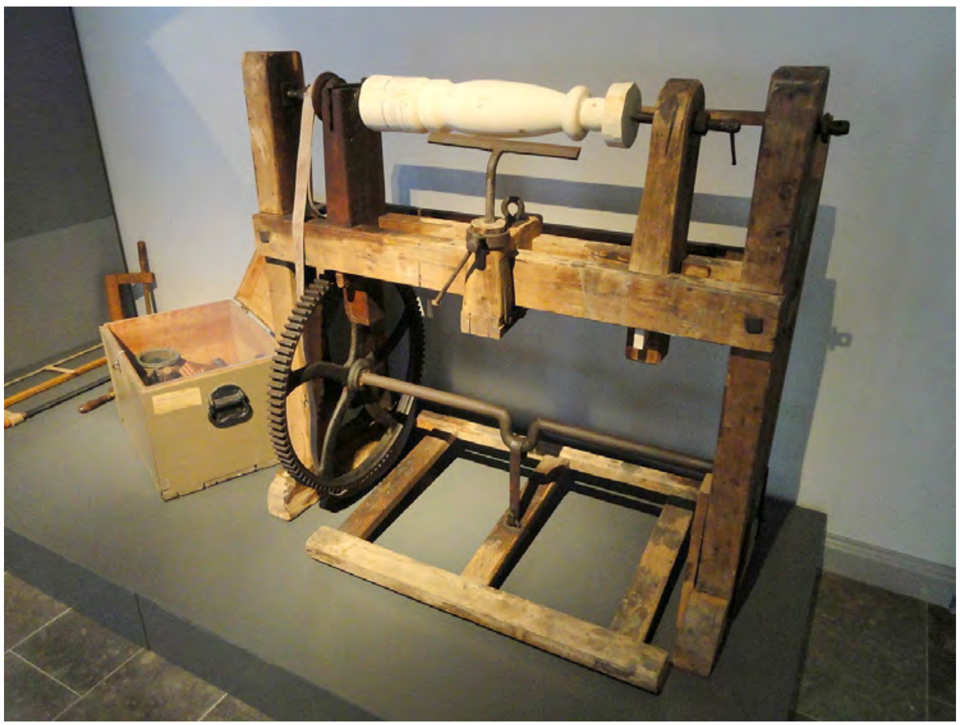


Figure 5.6-19 A wooden object shaped while spinning on this lathe takes the shape of a solid of revolution.

Figure 5.6-20 The flying disk is a solid of revolution, but it is not manufactured by revolving equipment.