

Worksheet 4.12 The Binomial Theorem

Section 1 BINOMIAL COEFFICIENTS AND PASCAL'S TRIANGLE

We wish to be able to expand an expression of the form $(a + b)^n$. We can do this easily for $n = 2$, but what about a large n ? It would be tedious to manually multiply $(a + b)$ by itself 10 times, say. There are two methods of expanding an expression of this type without doing all the multiplications involved. The first method we will look at is called Pascal's triangle. The first 5 rows of Pascal's triangle are shown:

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & & 1 & 1 \\ & & & & 1 & 2 & 1 \\ & & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{array}$$

Pascal's triangle is particularly useful when dealing with small n . The triangle is easy to remember as each entry is the sum of the two right and left entries on the line above, and the sides are always one. Thus for the second entry of line five we get

$$\begin{array}{cc} 1 & 3 \\ & \searrow \swarrow \\ & 4 \end{array}$$

The entries in Pascal's triangle tell us the coefficients when we expand expansions like $(a + b)^n$. Look at $(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2 = 1a^2 + 2ab + 1b^2$. Notice that the coefficients in front of each term 1 2 1 appear in the 3rd line of Pascal's triangle.

What about $(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$. Notice that the coefficients come from the 4th line of Pascal's triangle as 1 3 3 1.

You may also notice that as the power of a decreases, the power of b increases. This phenomenon can be generalised.

For the expansion of $(a + b)^n$ we need the $(n + 1)^{th}$ line of Pascal's triangle. The first term in the line has an $a^n b^0$ and then the powers of a decrease and the powers of b increase, so the second term will be $a^{n-1} b^1$ and so on, until the last term has $a^0 b^n$.

Example 1 : Expand $(a + b)^5$. We use the 6th line of Pascal's triangle to obtain

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Notice that the powers of a and b in each term always add to n , where n is the power to which $(a+b)$ is raised. In the above example we can see the power of a and b in each term always adds to 5.

Example 2 : Expand $(1 + x)^4$. Using the fifth row of Pascal's triangle:

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

If you wish to use Pascal's triangle on an expansion of the form $(ax + b)^n$, then some care is needed. The $(n + 1)$ th row is the row we need, and the 1st term in the row is the coefficient of $(ax)^n b^0$. The second term in the row is the coefficient of $(ax)^{n-1} b^1$. The last term in the row - the $(n + 1)$ th term - is the coefficient of $(ax)^0 b^n$. Care should be taken when minus signs are involved.

Example 3 : Expand the expression $(ax + b)^3$. The 4th row of Pascal's triangle is 1 3 3 1. So

$$\begin{aligned} (ax + b)^3 &= (ax)^3 + 3(ax)^2 b + 3(ax)^1 b^2 + b^3 \\ &= a^3 x^3 + 3a^2 b x^2 + 3a b^2 x + b^3 \end{aligned}$$

Notice that the powers of a and b in each term always add to give the power of the expansion. This is always the case.

Example 4 : Expand the expression $(2x - 3)^3$. Using the 4th row of pascals triangle:

$$\begin{aligned} (2x - 3)^3 &= (2x)^3 + 3(2x)^2(-3) + 3(2x)^1(-3)^2 + (-3)^3 \\ &= 8x^3 - 36x^2 + 54x - 27 \end{aligned}$$

Example 5 : What is the coefficient of x^2 in the expansion of $(x + 2)^5$? The 6th line of Pascal's triangle is 1 5 10 10 5 1. Remember when we expand $(a + b)^5$ the powers of a decrease as the powers of b increase and their powers will add to 5. In this example $a = x$ and $b = 2$. The term with an x^2 in it will be of the form $a^2 b^3$ - the fourth term in the expansion. Hence we use the fourth number from our line in Pascal's triangle as the coefficient in front of this term. So the term will look like $10a^2 b^3$. Since $a = x$ and $b = 2$ and $2^3 = 8$ we see that $10a^2 b^3 = 10x^2 2^3 = 80x^2$. Thus, the coefficient of x^2 is 80.

Example 6 : Find the constant term (the term that is independent of x) in the expansion of $(x - 2)^5$. The constant term is the last term, and is $(-2)^5$. Notice that the minus sign is important.

Example 7 : Find the 4th term in the expansion of $(2x - 3)^5$. The 4th term in the 6th line of Pascal's triangle is 10. So the 4th term is

$$10(2x)^2(-3)^3 = -1080x^2$$

The 4th term is $-1080x^2$.

The second method to work out the expansion of an expression like $(ax + b)^n$ uses binomial coefficients. This method is more useful than Pascal's triangle when n is large.

Exercises:

1. Write the first 6 lines of Pascal's triangle.
2. Expand $(x + y)^4$ using Pascal's triangle.
3. What line of Pascal's triangle is 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1?
4. Complete this line of Pascal's triangle "1, 8, 28, 56, 70, 56, ...". Hence also write the next line of Pascal's triangle.
5. Expand $(2a - 3)^5$ using Pascal's triangle.

Section 2 BINOMIAL THEOREM

Calculating coefficients in binomial functions, $(a + b)^n$, using Pascal's triangle can take a long time for even moderately large n . For example, it might take you a good 10 minutes to calculate the coefficients in $(x + 1)^8$. Instead we can use what we know about combinations.

Example 1 : What is the coefficient of x^7 in $(x + 1)^{39}$

To answer this, we think of it as a counting question. In the product of 39 copies of $(x + 1)$ we need to choose 7 x 's, and the order that they are chosen in does not matter. Thus the coefficient of x^7 is ${}^{39}C_7 = 15,380,937$.

So it turns out that the numbers in Pascals triangle, which we saw were coefficients in binomial expansions, are actually the numbers that come up in combinations, nC_k . For simplicity of writing we will define ${}^nC_k = \binom{n}{k}$.

Definition 2 : The binomial theorem gives a general formula for expanding all binomial functions:

$$\begin{aligned}(x + y)^n &= \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{r} x^{n-r} y^r + \cdots + \binom{n}{n} y^n,\end{aligned}$$

recalling the definition of the sigma notation from Worksheet 4.6.

Example 2 : Expand $(x + y)^8$

$$\begin{aligned}(x + y)^8 &= \binom{8}{0} x^8 + \binom{8}{1} x^7 y + \binom{8}{2} x^6 y^2 + \binom{8}{3} x^5 y^3 + \binom{8}{4} x^4 y^4 + \binom{8}{5} x^3 y^5 \\ &\quad + \binom{8}{6} x^2 y^6 + \binom{8}{7} x y^7 + \binom{8}{8} y^8 \\ &= x^8 + 8x^7 y + 28x^6 y^2 + 56x^5 y^3 + 70x^4 y^4 + 56x^3 y^5 + 28x^2 y^6 + 8xy^7 + y^8\end{aligned}$$

Example 3 : Expand $(2x - 3)^5$

$$\begin{aligned}(2x - 3)^5 &= \binom{5}{0} (2x)^5 + \binom{5}{1} (2x)^4 (-3) + \binom{5}{2} (2x)^3 (-3)^2 + \binom{5}{3} (2x)^2 (-3)^3 \\ &\quad + \binom{5}{4} 2x (-3)^4 + \binom{5}{5} (-3)^5 \\ &= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243\end{aligned}$$

Example 4 : Expand $\left(\frac{1}{x} + 2x^2\right)^4$

$$\begin{aligned}\left(\frac{1}{x} + 2x^2\right)^4 &= \binom{4}{0} \left(\frac{1}{x}\right)^4 + \binom{4}{1} \left(\frac{1}{x}\right)^3 2x^2 + \binom{4}{2} \left(\frac{1}{x}\right)^2 (2x^2)^2 + \binom{4}{3} \left(\frac{1}{x}\right) (2x^2)^3 \\ &\quad + \binom{4}{4} (2x^2)^4 \\ &= \frac{1}{x^4} + \frac{8}{x} + 24x^2 + 32x^5 + 16x^8\end{aligned}$$

Example 5 : Find the coefficient independent of x in

$$\left(x^3 + \frac{2}{x}\right)^{20}.$$

The binomial theorem tells us that

$$\left(x^3 + \frac{2}{x}\right)^{20} = \sum_{i=0}^{20} \binom{20}{i} x^{3i} \left(\frac{2}{x}\right)^{20-i} = \sum_{i=0}^{20} \binom{20}{i} x^{3i-(20-i)} 2^{20-i}.$$

So the power of x is $4i - 20$. We need to set this to zero to have the constant term, so we need $4i - 20 = 0 \rightarrow 4i = 20 \rightarrow i = 5$. Thus the coefficient is

$$\binom{20}{5} 2^{15} = 508,035,072.$$

Exercises:

1. Expand

(a) $(x^2 - 1)^4$

(b) $(x^3 - \frac{1}{x^2})^3$

2. Find the coefficients of x , x^2 and x^3 in $(x + 2)^5$.

3. Find the coefficients of x , x^2 and x^4 in $(x - 2)^7$.

4. Find the coefficients of x and x^{-9} in $(2x^3 - \frac{3}{x})^7$.

5. Find the exact value of $(1 - 0.1)^3$ without the use of a calculator. Confirm your answer with a calculator.

Exercises for Worksheet 4.12

1. (a) Show that ${}^5C_2 = {}^5C_3$
(b) Write down expansions of the following:
 - i. $(2x + 3y)^4$
 - ii. $\left(a + \frac{1}{a}\right)^6$
 - iii. $\left(\frac{a}{b} - \frac{b}{a}\right)^7$
 - iv. $(x^2 - 2)^5$
2. (a) Find the coefficients of
 - i. x^2 in $\left(x + \frac{1}{x}\right)^8$
 - ii. a^5b^4 in $\left(3a - \frac{b}{3}\right)^9$(b) Find the constant terms in
 - i. $\left(2x - \frac{1}{x^2}\right)^9$
 - ii. $\left(2x + \frac{1}{x}\right)^{10}$
3. Express $(1.1)^4$ as a binomial of the form $(a + b)^n$, and evaluate it.

Answers for Worksheet 4.12

Section 1

1. Not given
2. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
3. 11th line
4. 1, 8, 28, 56, 70, 56, 28, 8, 1 and 1, 9, 36, 84, 126, 126, 84, 36, 9, 1
5. $32a^5 - 240a^4 + 720a^3 - 1080a^2 + 810a - 243$

Section 2

1. (a) $1 - 4x^2 + 6x^4 - 4x^6 + x^8$
(b) $x^9 - 3x^4 + \frac{3}{x} - \frac{1}{x^6}$
2. ${}^5C_1 \times 2^4$, ${}^5C_2 \times 2^3$, ${}^5C_3 \times 2^2$
3. ${}^7C_1 \times (-2)^6$, ${}^7C_2 \times (-2)^5$, ${}^7C_4 \times (-2)^3$
4. ${}^7C_2 \times 2^2(-3)^5$, 0
5. 0.729

Exercises 4.12

1. (a) i. $16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$
ii. $a^6 + 6a^4 + 15a^2 + 20 + \frac{15}{a^2} + \frac{6}{a^4} + \frac{1}{a^6}$
iii. $\frac{a^7}{b^7} - \frac{7a^5}{b^5} + \frac{21a^3}{b^3} - \frac{35a}{b} + \frac{35b}{a} - \frac{21b^3}{a^3} + \frac{7b^5}{a^5} - \frac{b^7}{a^7}$
iv. $x^{10} - 10x^8 + 40x^6 - 80x^4 + 80x^2 - 32$
2. (a) i. 56 (b) i. -5376
ii. 378 ii. 8064
3. 1.4641