9.8 Graphing Rational Functions

Lets begin with a definition.

Definition: Rational Function
A rational function is a function of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where P and Q are polynomials.

An example of a simple rational function that we have seen before is $f(x) = \dfrac{1}{x}$.

As we can see, rational functions have limitations on their domains. The denominator can not be zero. So what happens when the denominator is zero? Lets investigate.

We will consider our function $f(x) = \dfrac{1}{x}$. Lets see what happens as we get closer and closer to zero from the positive and negative sides of zero. Here is a table of values.

x	f(x)
1	1
0.1	10
0.01	100
0.001	1000
0.0001	10,000

x	f(x)
-1	-1
-0.1	-10
-0.01	-100
-0.001	-1000
-0.0001	-10,000

As we can see, the functional values continue to increase as we get closer to zero from the positive side and they continue to become large negative from the negative side. We say that as x tends to zero, f(x) tends to infinity or negative infinity, respectively.

This gives us the graph of
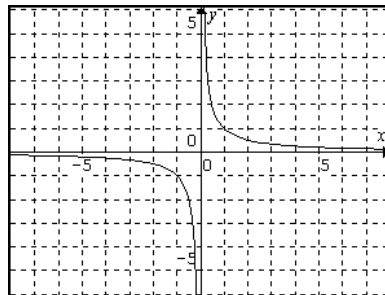
When this situation happens (on one side or the other, or both) we have what is called a <u>vertical asymptote</u>.

Property for Vertical Asymptotes
A rational function, which has no common factors in the numerator and denominator, has vertical asymptotes at all points where the denominator is zero.

So in order to find the vertical asymptotes we simply need to make sure there are no common factors and then find when the denominator is zero. These values would be the vertical asymptotes.

The next thing we notice about our function $f(x) = \dfrac{1}{x}$ is that as the values of x increase, the values of f(x) get closer to zero. And as the values become increasingly large negatives the values of f(x) get closer to zero as well.

This idea is one of a <u>horizontal asymptote</u>. A horizontal asymptote is any value that a function gets close to as the values of x get increasingly large in the positive or negative direction.

We use the following property to determine the horizontal asymptote. The proof of this property is reserved for precalculus.

Property for Horizontal Asymptotes

Let $f(x) = \dfrac{P(x)}{Q(x)}$ where the leading term of P is ax^n and the leading term of Q is bx^m. Then

    a.   If $n < m$, then the horizontal asymptote is at y = 0.
    b.   If $n > m$, then the function has no horizontal asymptote.

    c.   If $n = m$, then the horizontal asymptote is at $y = \dfrac{a}{b}$.

So we need only to compare the leading terms of the numerator and denominator to determine the horizontal asymptotes.

<u>Example 1</u>:

Determine all the asymptotes of $f(x) = \dfrac{3x^2 - 5x + 2}{x^2 - 5x - 6}$.

Solution:

First we need to factor the numerator and denominator to see if we can cancel any common factors.

$$f(x) = \frac{3x^2 - 5x + 2}{x^2 - 5x - 6}$$
$$= \frac{(3x - 2)(x - 1)}{(x - 6)(x + 1)}$$

Now we will start with the vertical asymptotes. The vertical asymptotes occur when the denominator is zero. So set the denominator equal to zero and solve. We get

$$x^2 - 5x - 6 = 0$$
$$(x - 6)(x + 1) = 0$$
$$x - 6 = 0 \quad x + 1 = 0$$
$$x = 6 \qquad x = -1$$

So the vertical asymptotes are at $x = 6$ and $x = -1$.

Now the horizontal asymptotes are found by using the property above. Since the degree of the numerator and denominator are the same, the horizontal asymptote occurs at the ratio of the leading coefficients. We get $y = \dfrac{3}{1} = 3$.

Next we need to discuss finding the intercepts of a rational function.

Recall, we find the y-intercept of an equation by letting x=0 and solve for y. We do the same thing in a rational function.

Also, recall, we find the x-intercepts of an equation by letting y=0 and solve for x. In a rational function however, this is the same as setting the numerator equal to zero and solving. The reason for this is that a fraction can only be zero when its numerator is zero.

Example 2:

Find the intercepts of $f(x) = \dfrac{3x^2 - 5x + 2}{x^2 - 5x - 6}$.

Solution:

We will start with the y-intercept. So we let x=0 and get

$$y = \frac{3(0)^2 - 5(0) + 2}{(0)^2 - 5(0) - 6}$$
$$= \frac{2}{-6}$$
$$= -\frac{1}{3}$$

For the x-intercepts we set the numerator equal to zero and solve.

$$3x^2 - 5x + 2 = 0$$
$$(3x - 2)(x - 1) = 0$$
$$3x - 2 = 0 \quad x - 1 = 0$$
$$x = \tfrac{2}{3} \qquad x = 1$$

The last thing we need in order to graph a rational function is a sign chart. A sign chart is a chart used to find where an equation is entirely positive and negative.

A sign chart tells us if the graph of a function is above the x-axis (positive) or below the x-axis (negative). Clearly, there are only a few ways in which a function can change from positive to negative. One way is to actually cross the x-axis, becoming an x –intercept, and the other way is if there is an asymptote.

Once we have a sign chart we can use this information with our intercepts and asymptotes to generate the graph of a rational function. Here are the steps involved in generating a sign chart.

| Creating a Sign Chart for a Rational function |
| --- |
| 1. Find and plot on a number line, all x-intercepts and vertical asymptotes. Use an open circle for the vertical asymptotes and a closed circle for the x-intercepts. |
| 2. Use the number line from step 1 to generate your intervals to be tested. We generally go from one asymptote or intercept to another to generate these intervals. |
| 3. Test any value in the intervals from step 2. The sign of this value will characterize the sign of the entire interval. |
| 4. Complete the sign chart by placing the sign of each interval directly above the interval on the graphed number line. |

Example 3:

Construct the sign chart for $f(x) = \dfrac{3x^2 - 5x + 2}{x^2 - 5x - 6}$.

Solution:

From examples 1 and 2 we know the vertical asymptotes and x-intercepts for

$f(x) = \dfrac{3x^2 - 5x + 2}{x^2 - 5x - 6}$. So we can put them on a number line as follows.

$$-1 \qquad \tfrac{2}{3} \qquad 1 \qquad 6$$
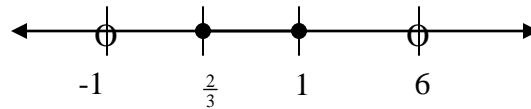
Note: We need to use open circle on the asymptotes since those values are not in the domain and thus are not defined, and filled circles on the x-intercepts since that is the spots at which the graph would actually be touching the x-axis.

So now we go from asymptote or intercept to asymptote to intercept to create our intervals. This would give us the following intervals: $\left(-\infty, -1\right), \left(-1, \tfrac{2}{3}\right), \left(\tfrac{2}{3}, 1\right), \left(1, 6\right), \left(6, \infty\right)$

We now test a value out of each interval. Since we are only concerned about the sign of the function in that interval, we only need to find if the test value makes the interval positive or negative.

For the first interval $\left(-\infty, -1\right)$, lets test –2.

$$f(-2) = \frac{3(-2)^2 - 5(-2) + 2}{(-2)^2 - 5(-2) - 6}$$
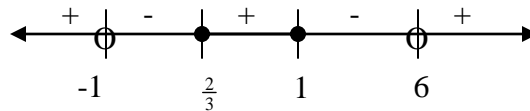$$= \frac{24}{8}$$
$$= \frac{+}{+} = +$$

So the first interval is positive.

For the next interval $\left(-1, \tfrac{2}{3}\right)$ we can test 0.

$$f(-2) = \frac{3(0)^2 - 5(0) + 2}{(0)^2 - 5(0) - 6}$$

$$= \frac{2}{-6}$$

$$= \frac{+}{-} = -$$

In a similar fashion we find that $\left(\frac{2}{3}, 1\right)$ is +, $(1, 6)$ is – and $(6, \infty)$ is +.

Summarizing this on the sign chart we get

Now all we have to do is take all of the information we have found and put it all on a graph.

Example 4:

Graph $f(x) = \dfrac{3x^2 - 5x + 2}{x^2 - 5x - 6}$.

Solution:

From example 1 we see that we have vertical asymptotes of $x = 6$, $x = -1$ and horizontal asymptote of $y = 3$. We start by graphing these on the rectangular grid with dashed lines. Then we plot the intercepts of $\left(\frac{2}{3}, 0\right), (1, 0)$ and $\left(0, -\frac{1}{3}\right)$. Finally we use the sign chart and the behavior around the asymptotes to finish the graph.
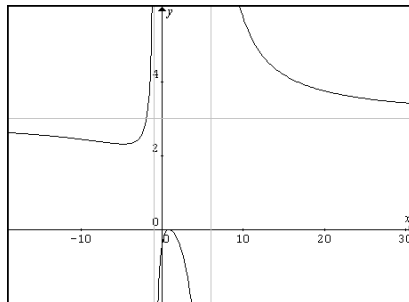
Plotting points can see the fact that the graph dips below the horizontal asymptote before getting close to it.

Lets now put all of this together in our last example to graph a rational function.

Example 5:

Graph $f(x) = \dfrac{x-1}{x^2 - x - 6}$.

Solution:

First we should put the function in factored form. $f(x) = \dfrac{x-1}{x^2 - x - 6} = \dfrac{x-1}{(x-3)(x+2)}$

From this we can see that the x-intercept is $x = 1$ and the y-intercept is $y = \frac{1}{6}$.
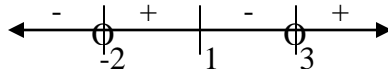
The vertical asymptotes occur at $x = -2$ and $x = 3$. Also, since the numerator has a smaller degree than the denominator, we have that the horizontal asymptote is $y = 0$.

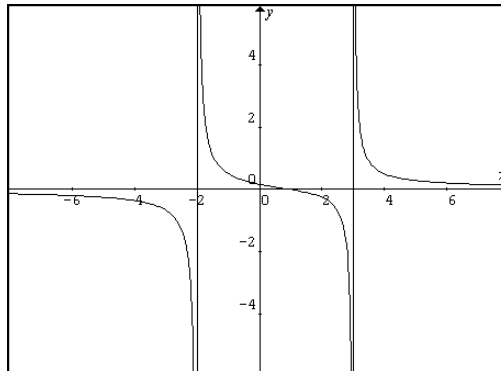Now using the x-intercepts and vertical asymptotes we can construct our sign chart.
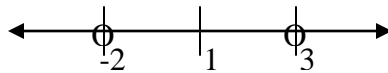
So our intervals to test are
$(-\infty, -2), (-2,1), (1,3), (3,\infty)$.
Testing a point in each interval will give us the following sign chart

So putting it all together we get the graph.

9.8 Exercises

Find all vertical and horizontal asymptotes of the following rational functions.

1. $f(x) = \dfrac{1}{x-1}$

2. $f(x) = \dfrac{2}{x-3}$

3. $f(x) = \dfrac{2x}{3x-1}$

4. $f(x) = \dfrac{x-1}{2x+1}$

5. $f(x) = \dfrac{x+5}{(x+1)(x-2)}$

6. $f(x) = \dfrac{2}{x(x+5)}$

7. $f(x) = \dfrac{2x-7}{x^2 - 2x - 8}$

8. $f(x) = \dfrac{x^2 - 4}{x^2 - 4x - 5}$

9. $f(x) = \dfrac{3x^3 + 12x}{2x^2 - 5x - 12}$

10. $f(x) = \dfrac{6x^2 + 4}{6x^2 + 5x - 6}$

11. $f(x) = \dfrac{2x^4 + 4x^2 + 1}{6x^3 - 24x^2 + 24x}$

12. $f(x) = \dfrac{25 - x^2}{3x^3 - 22x^2 - 16x}$

13. $f(x) = \dfrac{x}{x^4 - 1}$

14. $f(x) = \dfrac{2x^5}{x^3 + 8}$

15. $f(x) = \dfrac{x^3 - 1}{x^2 + x}$

Find all x- and y-intercepts of the following rational functions.

16. $f(x) = \dfrac{1}{x - 1}$

17. $f(x) = \dfrac{2}{x - 3}$

18. $f(x) = \dfrac{2x}{3x - 1}$

19. $f(x) = \dfrac{x - 1}{2x + 1}$

20. $f(x) = \dfrac{x + 5}{(x + 1)(x - 2)}$

21. $f(x) = \dfrac{x(x - 1)}{x + 5}$

22. $f(x) = \dfrac{(2x - 7)(x + 3)}{x^2 - 2x - 8}$

23. $f(x) = \dfrac{x^2 - 4}{x^2 - 4x - 5}$

24. $f(x) = \dfrac{3x^3 - 27x}{2x^2 - 2x + 12}$

25. $f(x) = \dfrac{x^3 + 4x}{6x^2 + 5x - 6}$

26. $f(x) = \dfrac{x^4 - 10x^2 + 9}{6x^3 - 24x^2 + 24x}$

27. $f(x) = \dfrac{x^4 - 29x^2 + 100}{3x^3 - 22x^2 - 16x}$

28. $f(x) = \dfrac{x^4 - 1}{x}$

29. $f(x) = \dfrac{2x^5 - 16x^4 - 40x^3}{x^3 + 8}$

30. $f(x) = \dfrac{x^6 - 1}{x^2 + 2x}$

Graph the following rational functions. Find all asymptotes and intercepts.

31. $f(x) = \dfrac{1}{x + 1}$

32. $f(x) = \dfrac{2}{x - 2}$

33. $f(x) = \dfrac{x}{x - 1}$

34. $f(x) = \dfrac{2x}{x - 2}$

35. $f(x) = \dfrac{x - 1}{x + 2}$

36. $f(x) = \dfrac{x - 2}{x - 1}$

37. $f(x) = \dfrac{3 - x}{x + 4}$

38. $f(x) = \dfrac{x + 2}{2 - x}$

39. $f(x) = \dfrac{x + 1}{(x - 2)(x - 1)}$

40. $f(x) = \dfrac{x}{(x + 1)(x + 3)}$

41. $f(x) = \dfrac{2x}{(2x - 1)(x + 1)}$

42. $f(x) = \dfrac{x - 2}{(x + 2)(x - 1)}$

43. $f(x) = \dfrac{(x-1)(x+2)}{(x-2)(x-3)}$ 44. $f(x) = \dfrac{x(x+1)}{(x-2)(x+2)}$ 45. $f(x) = \dfrac{x}{(x+2)^2}$

46. $f(x) = \dfrac{x^2}{(x+1)^2}$ 47. $f(x) = \dfrac{x}{x^2 - 2x - 3}$ 48. $f(x) = \dfrac{2x}{x^2 + 2x + 1}$

49. $f(x) = \dfrac{x+1}{x^2 + 2x - 3}$ 50. $f(x) = \dfrac{x^2 - 1}{x^2 + x}$ 51. $f(x) = \dfrac{x^2 - 6x + 8}{x^2 - 3x}$

52. $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - 4x + 3}$ 53. $f(x) = \dfrac{x^2 - 4x + 3}{x^2 + 2x + 1}$ 54. $f(x) = \dfrac{x^2 - 2x + 1}{x^2 - 2x - 3}$

55. $f(x) = \dfrac{2x^2 - 3x - 2}{x^2 - 3x - 4}$ 56. $f(x) = \dfrac{x^2 - x - 6}{2x^2 - 3x - 2}$ 57. $f(x) = \dfrac{2x^2 - 18}{x^2 - 4}$

58. $f(x) = \dfrac{4x^2 - 8x + 4}{x^2 - 4x + 5}$ 59. $f(x) = \dfrac{x-1}{x^3 - x}$ 60. $f(x) = \dfrac{2x^2 - 2}{x^3 - 4x}$

In the section we mentioned that the vertical asymptotes occurred where the denominator is zero, as long as the numerator and denominator have no common factors. In the case where they do have common factor it is a little different. For example $f(x) = \dfrac{(x-1)(x-1)}{x-1}$ clearly have a common factor. If we cancel it out we end up with $f(x) = x - 1,\ x \neq 1$. Since the domain is still all real numbers except 1 the graph of f is just the line y=x-1 but with a hole at x=1 since the function is not defined there. Use this idea to graph the following functions.

61. $f(x) = \dfrac{x^2 - 2x + 1}{x - 1}$ 62. $f(x) = \dfrac{x^2 - x}{x}$ 63. $f(x) = \dfrac{4x^2 + 4x - 3}{2x - 1}$

64. $f(x) = \dfrac{x^3 - x}{x^2 - x}$ 65. $f(x) = \dfrac{x+2}{x^2 - x - 6}$ 66. $f(x) = \dfrac{x-1}{x^2 + 4x - 5}$