QUADRATIC EQUATIONS

A **quadratic equation** is always written in the form of:

$$ax^2 + bx + c = 0 \qquad \text{where} \quad a \neq 0$$

The form $ax^2 + bx + c = 0$ is called the *standard form* of a quadratic equation.

Examples:

$x^2 - 5x + 6 = 0$ — This is a quadratic equation written in standard form.

$x^2 + 4x = -4$ — This is a quadratic equation that is not written in standard form but can be once we set the equation to: $x^2 + 4x + 4 = 0$.

$x^2 = x$ — This too can be a quadratic equation once it is set to 0. $x^2 - x = 0$ (standard form with c=0).

Solving Quadratic Equations by Square Root Property

When $x^2 = a$, where a is a real number, then your $x = \pm\sqrt{a}$

Examples:

$x^2 - 9 = 0$	$y^2 + 3 = 28$
$x^2 - 9 = 0$	$y^2 + 3 - 3 = 28 - 3$
$x^2 = 9$	$y^2 = 25$
$x = \pm\sqrt{9}$	$y = \pm\sqrt{25}$
$x = \pm 3$	$y = \pm 5$

Solving Quadratic Equations by Factoring

It can also be solved by factoring the equation. Remember to always check your solutions. You can use direct substitution of the solutions in the equation to see if the solutions satisfy the equation.

Examples:

$$x^2 - 5x + 6 = 0$$

$$(x-3)(x-2) = 0 \qquad \leftarrow \text{Factoring } x$$

$$x - 3 = 0 \qquad x - 2 = 0$$

$$x = 3 \qquad x = 2 \qquad \leftarrow \text{Set it equal to 0 and solve for } x$$

Now check if, $x = 3$ and $x = 2$ are the solutions of $x^2 - 5x + 6 = 0$

Check:

$$3^2 - 5(3) + 6 = 0 \qquad 2^2 - 5(2) + 6 = 0$$

$$9 - 15 + 6 = 0 \qquad 4 - 10 + 6 = 0$$

$$2x^2 + 7x - 4 = 0$$

$$(2x-1)(x+4) = 0$$

$$2x - 1 = 0 \qquad x + 4 = 0$$

$$2x = 1 \qquad x = -4$$

$$x = \frac{1}{2}$$

Another method of checking the solutions is by using one of the following statements:

The sum of the solutions = $-\frac{b}{a}$ *or* The product of the solutions = $\frac{c}{a}$
where *a*, *b*, and *c* are the coefficients in $ax^2 + bx + c = 0$.

Now we check if $x = \frac{1}{2}$ and $x = -4$ are the solutions of $2x^2 + 7x - 4 = 0$

Check: Using the sum of the solutions = $\frac{1}{2} + (-4) = -\frac{7}{2}$

Based on the original equation = $-\frac{b}{a} = -\frac{7}{2}$

Now by using the product of the solutions = $\frac{1}{2}(-4) = -2$

Based on the original equation = $\frac{c}{a} = \frac{-4}{2} = -2$

$1 + \frac{2}{x} - \frac{8}{x^2} = 0$ ← Rewrite in standard form by multiplying each side of the equation by x^2

$x^2 + 2x - 8 = 0$
$(x+4)(x-2) = 0$

$x + 4 = 0$ $\quad$ $x - 2 = 0$
$x = -4$ $\quad$ $x = 2$

Check: $1 + \frac{2}{-4} - \frac{8}{(-4)^2} = 0$ ← Solutions must be checked in the original equation to avoid any errors.

$1 - \frac{1}{2} - \frac{1}{2} = 0$

$1 + \frac{2}{2} - \frac{8}{2^2} = 0$

$1 + 1 - 2 = 0$

Solution Using the Quadratic Formula

Factoring is useful only for those quadratic equations which have whole numbers. When you encounter quadratic equations that can not be easily factored out, use the quadratic formula to find the value of x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Examples: $x^2 - 8 = -2x$

$x^2 + 2x - 8 = 0$ ← Rewrite in standard form, where $a = 1,\ b = 2,$ and $c = -8$

$x = \dfrac{-2 \pm \sqrt{4 - 4(1)(-8)}}{2(1)}$ ←Plug in numbers into the equation

$= \dfrac{-2 \pm \sqrt{36}}{2(1)}$

$= \dfrac{-2 \pm 6}{2}$

$= 2, -4$ ← The two rational solutions

$3x^2 - 13x + 4 = 0$

$x = \dfrac{-(-13) \pm \sqrt{(-13)^2 - 4(3)(4)}}{2(3)}$

$= \dfrac{13 \pm \sqrt{121}}{6}$

$= \dfrac{13 \pm 11}{6}$

$= \dfrac{24}{6}, \dfrac{2}{6} = 4, \dfrac{1}{3}$ ←The two rational solutions

In some cases you encounter repeated rational solutions. And to prove you have the right values you use the discriminant which gives you information about the nature of the solutions to the equation. Based on the expression $b^2 - 4ac$, which is under the radical in the quadratic formula it can be found in the equation $ax^2 + bx + c = 0$.

I. When the discriminant is equal to 0, the equation has repeated rational solutions.

Example: $x^2 - 2x + 1 = 0$

By using the discriminant $b^2 - 4ac = (-2)^2 - 4(1)(1) = 0$

$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$

$= \dfrac{2 \pm \sqrt{0}}{2}$

$x = 1,1$ ←Repeated rational solutions

II. When the discriminant is positive and a perfect square, the equation has two distinct rational solutions.

Example: $x^2 - 4x + 3 = 0$

By discriminant $b^2 - 4ac = (-4)^2 - 4(1)(3) = 4$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{4}}{2}$$

$x = 3,1$ $\leftarrow$ Two distinct rational solutions

III. When the discriminant is positive but not a perfect square, the equation has two irrational solutions.

Example: $x^2 + 4x - 6 = 0$

The discriminant $b^2 - 4ac = (4)^2 - 4(1)(-6) = 40$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-6)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{40}}{2}$$

$x = -2 \pm \sqrt{10}$ $\leftarrow$ Two irrational solutions

IV. When the discriminant is negative, the equation has two complex number solutions.

Example: $x^2 + 4x + 6 = 0$

The discriminant $b^2 - 4ac = (4)^2 - 4(1)(6) = -8$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(6)}}{2(1)}$$

$x = \dfrac{-4 \pm \sqrt{-8}}{2}$ $\leftarrow$ Two complex number solutions

Solution by Completing the Square

One more method of solving quadratic equations is by completing the square.

Example: Solve $x^2+6x+5=0$ by completing the square.

1) If the leading coefficient is not 1, use the multiplication (or division) property of equality to make it 1:

$x^2+6x+5=0$ $\leftarrow$ In this case the leading coefficient is already 1

2) Rewrite the equation by sending the constant to the right side of the equation:

$$x^2+6x+5=0$$
$$x^2+6x+5-5=0-5$$
$$x^2+6x=-5$$

3) Divide the numerical coefficient the middle term by 2, then square it, and add it to both sides of the equation, but leave the square form on the left side of the equation:

$$x^2+6x=-5$$
$$x^2+6x+(3)^2=-5+(3)^2$$
$$x^2+6x+(3)^2=-5+9$$
$$x^2+6x+(3)^2=4$$

Middle term coefficient = 6 $\qquad \frac{6}{2}=3\rightarrow(3)^2$

4) Once you found the squared number rewrite the equation as follows:

$x^2+6x+(3)^2=4$ $\leftarrow$ Bring down the variable x and put it inside the parentheses

$\leftarrow$ Use the sign of the middle term. In this case it is +.

$(x+3)^2=4$ $\leftarrow$ Write the squared number. In this case it is 3.

The resultant binomial is $(x+3)^2=4$

5) Using the square root property clear the term.

$\sqrt{(x+3)^2}=\pm\sqrt{4}$ $\leftarrow$ The square root of a squared term is the term by itself.

$x+3=\pm 2$

6) Solve for the variable x.

$x+3=\pm 2$ $\leftarrow$ The $\pm$ notation is used because the square root can have both positive and negative answers.

$x+3=2$	$x+3=-2$
$x=2-3$	$x=-2-3$
$x=-1$	$x=-5$

$(-1+3)^2=(2)^2=4$ And $(-5+3)^2=(-2)^2=4$

7) Check your solution.

$$x=-1$$
$$x^2+6x+5=0$$
$$(-1)^2+6(-1)+5=0$$
$$1-6+5=0$$
$$0=0$$

← Both solutions are true:

$$x=-5$$
$$x^2+6x+5=0$$
$$(-5)^2+6(-5)+5=0$$
$$25-30+5=0$$
$$0=0$$

Let’s keep practicing with one more.

Example:

$$4x^2-2x-5=0$$
$$\frac{1}{4}\left(4x^2-2x-5\right)=\frac{1}{4}(0)$$

← One way to make the leading coefficient 1 is by multiplying both sides of the equation by $\frac{1}{4}$

$$x^2-\frac{1}{2}x-\frac{5}{4}=0$$

← Move the constant to the right side of the equation

$$x^2-\frac{1}{2}x=\frac{5}{4}$$
$$x^2-\frac{1}{2}x+\left(\frac{1}{4}\right)^2=\frac{5}{4}+\left(\frac{1}{4}\right)^2$$

← Divide the middle term coefficient by 2, square it, and add it to both sides of the equation:

$$\frac{1}{2}\div\frac{2}{1}=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}\ \rightarrow\ \left(\frac{1}{4}\right)^2$$
$$x^2-\frac{1}{2}x+\left(\frac{1}{4}\right)^2=\frac{5}{4}+\frac{1}{16}$$
$$x^2-\frac{1}{2}x+\left(\frac{1}{4}\right)^2=\frac{21}{16}$$

$$\left(x-\frac{1}{4}\right)^2=\frac{21}{16}$$

← Write the squared number in binomial form.

$$\sqrt{\left(x-\frac{1}{4}\right)^2}=\pm\sqrt{\frac{21}{16}}$$

← Find the square root of both sides and don’t forget the $\pm$ sign.

$$x-\frac{1}{4}=\pm\frac{\sqrt{21}}{\sqrt{16}}$$

← Send the other number to the right side of the equation.

$$x - \frac{1}{4} = \pm\frac{\sqrt{21}}{4}$$ ← Try to solve it using the square root. If not possible leave it in radical form.

$$x = \pm\frac{\sqrt{21}}{4} + \frac{1}{4}$$ ← Solve for *x*.

$$x = \frac{1 \pm \sqrt{21}}{4}$$ ← Final answer.

QUADRATIC EQUATIONS – EXERCISES

Solve each of the following equations by the method of your choice and check your solutions.

1. $x^2 - 2x + 1 = 0$
2. $x^2 + 9x + 20 = 0$
3. $3x^2 - 5x - 12 = 0$
4. $6x^2 + 9x - 6 = 0$
5. $x^2 + 3x - 28 = 0$
6. $3x^2 - 2x = 2x + 7$
7. $4x^2 - 12x = 16$
8. $x^2 + 3x = 0$
9. $3 + \frac{1}{x} = \frac{10}{x^2}$
10. $3y^2 - y - 4 = 0$
11. $y^2 + 2y + 1 = 0$
12. $x^2 - 2x - 8 = 0$
13. $x^2 + 4 = 0$
14. $x^2 + x = -1$
15. $9y^2 + 6y - 8 = 0$
16. $y^2 - 25 = 0$
17. $6y^2 - 13y + 6 = 0$
18. $x - \frac{4}{3x} = \frac{-1}{3}$
19. $x - \frac{4}{x} = \frac{21}{5}$
20. $3 + \frac{5}{2x} = \frac{1}{x^2}$
21. $4 - \frac{1}{x} = \frac{3}{x^2}$
22. $8x = x^2$
23. $(x-5)(x+8) = -20$
24. $(x+6)(x-3) = 10$
25. $\frac{2}{x+5} - \frac{x}{x-5} = 1$

QUADRATIC EQUATIONS – ANSWER TO EXERCISES

1. $x^2 - 2x + 1 = 0$
   $(x-1)(x-1) = 0$

   $x - 1 = 0 \qquad x - 1 = 0$

   $x = 1 \qquad x = 1$

2. $x^2 + 9x + 20 = 0$
   $(x+5)(x+4) = 0$

   $x + 5 = 0 \qquad x + 4 = 0$

   $x = -5 \qquad x = -4$

3. $3x^2 - 5x - 12 = 0$
   $(3x+4)(x-3) = 0$

   $3x + 4 = 0 \qquad x - 3 = 0$

   $x = -\frac{4}{3} \qquad x = 3$

4. $6x^2 + 9x - 6 = 0$
   $3(2x^2 + 3x - 2) = 0$
   $3(2x-1)(x+2) = 0$
   $2x - 1 = 0 \qquad x + 2 = 0$

   $x = \frac{1}{2} \qquad x = -2$

5. $x^2 + 3x - 28 = 0$
   $(x+7)(x-4) = 0$

   $x + 7 = 0 \qquad x - 4 = 0$

   $x = -7 \qquad x = 4$

6. $3x^2 - 2x = 2x + 7$
   $3x^2 - 2x - 2x - 7 = 0$
   $3x^2 - 4x - 7 = 0$
   $(3x-7)(x+1) = 0$

   $3x - 7 = 0 \qquad x + 1 = 0$

   $x = \frac{7}{3} \qquad x = -1$

7. $4x^2 - 12x = 16$
   $4x^2 - 12x - 16 = 0$
   $4(x^2 - 3x - 4) = 0$
   $4(x-4)(x+1) = 0$

   $x - 4 = 0 \qquad x + 1 = 0$

   $x = 4 \qquad x = -1$

8. $x^2 + 3x = 0$
   $x(x+3) = 0$

   $x = 0 \qquad x + 3 = 0 \qquad x = -3$

9. $3 + \frac{1}{x} = \frac{10}{x^2}$
   $3x^2 + x = 10$
   $3x^2 + x - 10 = 0$

   $(3x-5)(x+2) = 0$

   $3x - 5 = 0 \qquad x + 2 = 0$

   $x = \frac{5}{3} \qquad x = -2$

10. $3y^2 - y - 4 = 0$

    $(3y-4)(y+1) = 0$

    $3y - 4 = 0 \qquad y + 1 = 0$

    $y = \frac{4}{3} \qquad y = -1$

11. $y^2+2y+1=0$

$(y+1)^2=0$

$y+1=0$

$y=-1,-1$

12. $x^2-2x-8=0$

$(x-4)(x+2)=0$

$x-4=0 \qquad x+2=0$

$x=4 \qquad x=-2$

13. $x^2+4=0$

$$x=\frac{-0\pm\sqrt{0-4(1)(4)}}{2}$$

$$x=\pm\frac{\sqrt{-16}}{2}$$

14. $x^2+x=-1$

$x^2+x+1=0$

$$x=\frac{-1\pm\sqrt{1-4(1)(1)}}{2(1)}$$

$$x=\frac{-1\pm\sqrt{-3}}{2}$$

15. $9y^2+6y-8=0$

$$x=\frac{-6\pm\sqrt{36-4(9)(-8)}}{2(9)}$$

$$=\frac{-6\pm\sqrt{324}}{18}$$

$$=\frac{-6\pm18}{18}$$

$$x=\frac{6+18}{18} \qquad x=\frac{6-18}{18}$$

$$x=\frac{2}{3} \qquad x=\frac{-4}{3}$$

16. $y^2-25=0$

$(y-5)(y+5)=0$

$y-5=0 \qquad y+5=0$

$y=5 \qquad y=-5$

17. $6y^2-13y+6=0$

$(3y-2)(2y-3)=0$

$3y-2=0 \qquad 2y-3=0$

$$y=\frac{2}{3} \qquad y=\frac{3}{2}$$

18. $x-\frac{4}{3x}=-\frac{1}{3}$

$3x^2-4=-x$

$3x^2+x-4=0$

$(3x+4)(x-1)=0$

$3x+4=0 \qquad x-1=0$

$$x=-\frac{4}{3} \qquad x=1$$

19. $x-\frac{4}{x}=\frac{21}{5}$

$$5x^2-20=21x$$
$$5x^2-21x-20=0$$
$$(5x+4)(x-5)=0$$
$$5x+4=0 \qquad x-5=0$$
$$x=-\frac{4}{5} \qquad x=5$$

20. $3+\frac{5}{2x}=\frac{1}{x^2}$

$$6x^2+5x=2$$
$$6x^2+5x-2=0$$
$$x=\frac{-5\pm\sqrt{25-4(6)(-2)}}{12}$$
$$x=\frac{-5\pm\sqrt{73}}{12}$$

21. $4-\frac{1}{x}=\frac{3}{x^2}$

$$4x^2-x=3$$
$$4x^2-x-3=0$$
$$(4x+3)(x-1)=0$$
$$4x+3=0 \qquad x-1=0$$
$$x=-\frac{3}{4} \qquad x=1$$

22. $8x=x^2$

$$x^2-8x=0$$
$$x(x-8)=0$$
$$x=0 \qquad x-8=0$$
$$x=8$$

23. $(x-5)(x+8)=-20$

$$x^2+3x-40+20=0$$
$$x^2+3x-20=0$$
$$x=\frac{-3\pm\sqrt{9-4(1)(-20)}}{2}$$
$$x=\frac{-3\pm\sqrt{89}}{2}$$

24. $(x+6)(x-3)=10$

$$x^2+3x-18-10=0$$
$$x^2+3x-28=0$$
$$(x+7)(x-4)=0$$
$$x+7=0 \qquad x-4=0$$
$$x=-7 \qquad x=4$$

25. $\frac{2}{x+5}-\frac{x}{x-5}=1$

$$2(x-5)-x(x+5)=x^2-25$$
$$2x-10-x^2-5x-x^2+25=0$$
$$-2x^2-3x+15=0$$
$$2x^2+3x-15=0$$
$$x=\frac{-3\pm\sqrt{9-4(2)(-15)}}{4} \quad x=\frac{-3\pm\sqrt{129}}{4}$$