Probabilities of Poker Hands with Variations

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Poker is one of the many games involving the use of a 52-card deck of playing cards. The 52 cards are categorized by 13 ranks from Two through Ace (Aces can be counted as both higher than King and lower than Two when needed, but can only count as one at a time in a hand), and by four suits: diamonds, hearts, spades, and clubs. In the game of poker, players attempt to assemble the best five-card hand according to the definitions of each hand that can be made.

There are ten hands that can be made:

1) Royal Flush – all five cards are of the same suit and are of the sequence 10 – J – Q – K – A

2) Straight Flush – all five cards are of the same suit and are sequential in rank (note that a royal flush is simply the highest-ranked straight flush)

3) Four-of-a-Kind (which will be abbreviated in this paper as 4OAK) – a hand where four cards are all of the same rank

4) Full House – a hand consisting of one pair and a three-of-a-kind of a different rank than the pair

5) Flush – all five cards are of the same suit but not all sequential in rank

6) Straight – all five cards are sequential in rank but are not all of the same suit

7) Three-of-a-Kind (which will be abbreviated as 3OAK) – a hand where three cards are all of the same rank and the other two are each of different ranks from the 3OAK and each other

8) Two Pair – two pairs of two cards of the same rank (the ranks of each pair are different in rank, obviously, to avoid a 4OAK)

9) One Pair – only two cards of the five are of the same rank with the other three cards all having different ranks from each other and from that of the pair

10) High Card – a hand in which no better hand was made (i.e., one in which each card is of a different rank than any other card and not all five are of the same suit or sequential in rank

Poker games have many variations, some of which will be investigated here. One such variation is "stud" poker in which a player must hold all the cards he/she is given. This is opposed to "draw" poker in which a player can draw any number of replacement cards after being dealt an initial five in the attempt to improve his/her hand. Texas Hold em is another variation in which each player is only dealt two cards to themselves, but through the course of the betting rounds a total of five cards are dealt as "community" cards that any player can use with any combination of their two to make the best five-card hand possible. Other variations include the use of jokers and wild cards. In this paper I will derive the probabilities of being dealt one of the given hands in five-card stud poker and how those probabilities change when jokers and wild cards are included. I will also analyze Texas Hold em and derive the probability of a given hand winning throughout the course of a few example games.

Five-Card Stud

In five-card stud each player is dealt five cards to make the best five-card hand possible. Since there are 52 cards in the deck, then there are ${}_{52}C_5$[1] = 2,598,960 possible combinations of five-card hands possible. I will evaluate the numbers of hands in the typical order of rank of each hand, starting with straight flushes (since a royal flush is just the highest-ranked straight flush I will include it in the discussion of straight flushes, but give it no additional importance).

Straight Flush

To have a straight flush the hand must consist of all five cards being of the same suit and all in numerical order. There are 10 possible sequences: A – 5, 2 – 6, … , 9 – K, and 10 – A. Since there are 4 suits, then the number of straight flushes possible is just 10 * 4 = 40, with the highest four (each a straight flush 10 – A of one of the four suits) being royal flushes.

Four-of-a-Kind (4OAK)

To have a 4OAK the hand must include all of the cards of one of the 13 available ranks plus one additional card. It doesn't matter what the last card is. There is only ${}_4C_4$ = 1 combination of all four cards of one rank, and there will be 48 remaining cards left to choose from after the 4OAK is obtained, so there are 13 * ${}_4C_4$ * 48 = 624 possible fours-of-a-kind.

Full House

Since a full house has the form of one pair plus a three-of-a-kind then there are 13 * 12 = 78 choices for the ranks of the pair and the 3OAK (note that I don't need to remove permutations from the choices because there is a difference in which of the pair or 3OAK gets which rank. For example a full house consisting of two 4s and three 9s is different than one consisting of two 9s and three 4s). There are ${}_4C_2$ = 6 choices for the pair in its rank and ${}_4C_3$ = 4 choices for the 3OAK. Therefore there are 12 * 13 * ${}_4C_2$ * ${}_4C_3$ = 3744 possible full houses.

Flush

A hand that is a flush must consist of all five cards being of the same suit. Each of the four suits has ${}_{13}C_5$ = 1287 possible five-card hands that are all of the same suit. However, some of those combinations are also straight flushes. Using a Venn diagram can help to visualize the overlapping of the sets. The 40 straight flushes must be removed from the count. Thus there are 4 * ${}_{13}C_5$ – 40 = 5108 possible flushes.

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[1] This is the notation I will use for the mathematical choose operation, nCr, which indicates the number of subsets of size r that can be formed from a set of n distinct objects. nCr = n!/[r!(n-r)!].

Straight

A hand that is a straight must consist of five cards sequential in rank, but with all five not all of the same suit. Using similar arguments from straight flushes and flushes: there are 10 sequences; there are 4 choices for the particular card in each rank. Thus there are $4^5 = 1024$ possible ways to choose the cards in each sequence. Taking away the 40 straight flushes results in the number of straights being $10 * 4^5 – 40 = 10{,}200$.

Three-of-a-Kind (3OAK)

This hand must consist of three cards being of the same rank with the other two not improving the hand. There are 13 ranks to choose from for the 3OAK and ${}_4C_3$ combinations of 3OAKs within each rank. There are (48 * 44)/2 possible choices for the last two cards (here I had to divide by 2, which is really 2! in order to remove the permutations that would double the count. In poker the order in which the cards appear does not matter). Thus there are $13 * {}_4C_3 * (48 * 44)/2 = 54{,}912$ possible 3OAKs.

Two Pair

There are ${}_{13}C_2$ ways to choose the two ranks for the two pair and ${}_4C_2$ ways to choose the pair in each rank. There are 44 cards possible for the fifth card so as not to improve the hand. Thus there are ${}_{13}C_2 * ({}_4C_2)^2 * 44 = 123{,}552$ possible two pair hands.

One Pair

Similar to arguments for previous hands there are 13 ranks to choose from for the pair and ${}_4C_2$ possible pairs per rank, plus (48 * 44 * 40)/6 ways to choose the other three cards (again to remove permutations and keep only combinations I must divide by 3!, the number of permutations of the three cards). This leaves $13 * {}_4C_2 * (48 * 44 * 40)/6 = 1{,}098{,}240$ possible one pair hands.

High Card

There are two ways to derive the number of high-card hands. One is by realizing that the set of high-card hands is the complement to the set of all other hands. That means the number of high card hands is 2598960 – 40 – 624 – 3744 – 5108 – 10200 – 54912 – 123552 – 1098240 = 1,302,540.

The other way is to manually derive this number by realizing that to make a high card hand the hand must consist of all five cards being unpaired, non-sequential in rank, and not all of the same suit. The product (52 * 48 * 44 * 40 * 36)/5! accounts for all hands involving some cards having the same rank (i.e., one pair, two pair, 3OAK, 4OAK, and full house). The rest must simply be subtracted off. This leaves (52 * 48 * 44 * 40 * 36)/120 – 40 – 5108 – 10200 = 1,302,540 high-card hands. Since the two methods each gave the same number then that is reason to believe the counting is correct.

The following table lists, for each hand, the number and probability of a given hand.

Five-Card Stud (Natural) Probabilities		
Hand	**Number**	**Probability**
Straight Flush[2]	40	0.00002
Four-of-a-Kind	624	0.00024
Full House	3744	0.00144
Flush	5108	0.00197
Straight	10,200	0.00393
Three-of-a-Kind	54,912	0.02113
Two Pair	123,552	0.04754
One Pair	1,098,240	0.42257
High Card	1,302,540	0.50118
Total	2,598,960	1.00005[3]

Note from this table that it isn't until you get down to the three-of-a-kind hand that the probability for any hand becomes significant. This is just a show of how improbable it is to deal one of the higher hands from a simple five-card deal. Also notice how the number (probability) of a hand increases (decreases) as you move from higher ranked to lower ranked hands. This is the idea behind ranking the individual hands. A hand that is less likely to occur is given a higher rank than one that is more likely to occur. A theme for the rest of this paper will be to use this theory to rank the hands in other variations and see if and how they change. However, a quick digression to the variation of Texas Hold em will precede the discussion of wild cards.

Texas Hold em

In the poker variation of Texas Hold em each player is initially dealt two cards (called "hole cards") and through the course of betting rounds five cards are dealt on the table that no single player owns outright, but that can be used by any player in the game, in conjunction with the player's own two cards, to make the best five-card poker hand possible. The five cards that are dealt on the table, called "community cards", are dealt out in the sequence of three-one-one. This means that after the first betting round three community cards are dealt (this is called "the flop"), then after the next round one more community card is dealt (called "the turn"), and finally one more community card is dealt after another betting round (called "the river"). There is one more round of betting after the river is dealt. After that, each player that is still in the game shows their hand and the one with the best five-card hand wins.

The mathematical approach to analyzing this game is to determine, given a certain setup for a game of Texas Hold em, the probability that any given player will win. I will analyze two examples: the first being a simple two-player game and the second a more

[2] Royal flushes are included in the straight flushes. The number and probability of a royal flush alone is 4 and 0.0000015.

[3] The error in the sum of the probabilities is due to rounding. In similar tables to come, the symbol * will indicate the same thing.

complicated four-player game. I will assume at every point during each game that the player is taking the best possible five-card hand.

Two-Player Texas Hold em

Two players are involved in a game of Texas Hold em. I will call them Player 1 and Player 2. The two hole cards dealt to each player are:

Player 1	*Player 2*
Kd Qc	*Ad 9c*

The lower-case letters next to each card indicate the suit of the card (d-diamonds, c-clubs, h-hearts, and s-spades). Although it is possible to calculate the probability of each player winning at this point, the calculation would be quite complicated and tedious, and since neither player can even make a five-card hand yet, it is more useful to see the flop before making an analysis. So let's see it.

The flop is: Ah Jd 5h

Each player can now make a five-card hand.

Player 1	*Player 2*
Ah Kd Qc Jd 5h	*Ah Ad Jd 9c 5h*

We can see that Player 2 has a pair of Aces while Player 1 has a high card Ace and a straight draw (Player 1 has four cards needed to make a straight and only needs one more card to get it. Whenever the word "draw" is added after a hand it means that the player with it only needs one more card in order to complete the hand referred to.) Player 2 thus has the best hand and it is the burden of Player 1 to get the necessary cards in order to win. I can now make my first analysis of the probabilities. First of all note that since seven cards have been dealt and two more will be dealt still then there are ${}_{45}C_2 = 990$ ways for the last two cards to be dealt. Therefore each player shares a certain number of those 990 two-card combinations that will give them the win. There are three possible outcomes in this game: Player 1 wins, Player 2 wins, or they tie. It may take a minute for the reader to determine this, but there is no combination of the last two cards that would give each player the same hand and result in a tie. Therefore the chance of a tie is 0. Since Player 2 has a better hand at this point then he/she will win unless the particular combinations of the last two cards that Player 1 needs to come out do so. Thus it is easiest to calculate the probabilities remaining by determining what chance Player 1 has of drawing the cards needed to improve his/her hand to win.

It may take a few minutes to analyze in the reader's head, but the possibilities are best summarized in the following table:

Possible Outcome	Player 1 wins	Player 2 wins	Tie
If the last two cards are…	10, Any K, Q K, K Q, Q	Else i.e., 1 – P(Player 1 wins)	Not possible

There are three kings, three queens, and four tens left in the deck. Therefore the number of combinations of each type is:

- 10, Any: 4 tens * 41 other non-ten cards + ${}_4C_2$ ten-ten combinations = 170 (Note: it would be incorrect to simply calculate 4 * (45 – 1 other cards) = 176 because within that count are each ten-ten combination counted twice. Thus to correct I would have to remove the six ten-ten combinations I over counted, or just remove them from the initial count and then add them back in as I did.)
- K, Q: 3 kings * 3 queens = 9
- K, K: ${}_3C_2$ ways to choose 2 of the remaining 3 kings = 3
- Q, Q: ${}_3C_2$ ways to choose 2 of the remaining 3 queens = 3

This brings the total number of two-card combinations that give Player 1 the win as 185, and a probability of winning of 185/990 = 0.187. Thus the number for Player 2 is 990 – 185 = 805, and the probability of winning is 805/990 = 0.813. Player 2 has nearly 4.4:1 odds on beating Player 1. Let's move on.

The turn card is: Qd

The hands are now:

<u>*Player 1*</u>	<u>*Player 2*</u>
Qc Qd Ah Kd Jd	*Ah Ad Qd Jd 9c*

Player 1 drew one of the cards needed to improve his/her hand. The queen dealt on the turn gave Player 1 a pair of Queens against Player 2's pair of Aces. Recalling the possible combinations of cards that would give Player 1 the best hand means that he/she will win if the next card is a king, queen, or a ten; Player 2 will win if the next card is anything else. Since eight cards have been dealt there are 44 left in the deck as possible river cards. Therefore the number of cards that will give Player 1 the win if dealt is:
4 tens + 3 kings + 2 queens = 9 cards; for a probability of 9/44 = 0.205. Player 2 has the other 35 cards for the win and a probability of 35/44 = 0.795.

The final card drawn is the eight of spades. Since this is not one of the cards Player 1 needed to obtain a better hand, then, given Player 2 does not fold his/her hand, he/she will win.

This was a rather simple example. Let's now consider a more complicated game.

Four-Player Texas Hold em

There are four players in this example game of Texas Hold em. The hole cards each player is dealt are:

Player 1	*Player 2*	*Player 3*	*Player 4*
Ad 7d	*9d 2h*	*10h 10c*	*4c 8d*

We can see that Player 3 has the best hand with a pair of Tens (some more counting tells us that there are ${}_{52}C_2$ = 1326 possible two-card combinations that can be dealt, and there are only 13 * ${}_{4}C_2$ = 78 pairs within those combinations. Thus the chances of being dealt a pair for the hole cards is 78/1326 = 0.059). Suppose the betting causes Player 4 to fold before the flop. Let's see the flop.

The flop is: Js Qs 9h

Each player now can make a five-card hand. They are:

Player 1	*Player 2*	*Player 3*
Ad Qs Js 9h 7d	*9h 9d Qs Js 2h*	*10h 10c Qs Js 9h*

Player 3 still has the best hand but Player 2 is close behind with a pair of Nines. Simple observation shows us that Player 1 will make a straight if a king and ten are the next two cards dealt, and Player 3 will make a straight if the next two cards include a king and any non-ten card. Much like the previous example, below is a table summarizing the possible outcomes and requirements to meet each outcome (note that with three players, there are four possible ways a tie can occur):

Possible Outcome	Player 1 wins	Player 2 wins	Player 3 wins	Tie
If the last two cards are…	A, 7 A, A 7, 7	9, 9 9, 2 2, 2 2 and ~(Q, J, or 10)	Else	K, 10 (all 3 tie) 10, 8 (Players 2 and 3 tie)

It is the burden of Players 1 and 2 to improve their hands to win. Player 3 wins if neither player can improve their and there is no tie. Note that a tie between Players 1 and 3 and between Players 1 and 2 is not possible. Since 11 cards have been dealt there are ${}_{41}C_2$ = 820 possible combinations of the last two cards to be dealt. Let's see how they break down.

For Player 1, there are three aces and three sevens left. Thus the combinations for the hands are:

- A, 7: 3 * 3 = 9
- A, A: ${}_{3}C_2$ = 3
- 7, 7: ${}_{3}C_2$ = 3

The total number and probability of Player 1 winning is (9 + 3 + 3)/820 = 15/820 = 0.018.

For Player 2, there are two nines and three twos left along with three queens, three jacks, and two tens. The number of combinations that result in Player 2 winning is:

- 9, 9: ${}_2C_2 = 1$
- 9, 2: 3 * 2 = 6
- 2, 2: ${}_3C_2 = 3$
- 2 and ~(Q, J, or 10) = 3 * 32 non-ten, queen, or jack cards = 96

The total number and probability of Player 2 winning is (96 + 1 + 6 + 3)/820 = 106/820 = 0.129.

There are four kings and two tens left, so the chance of a king-ten combination giving all three players a tie is (4 * 2)/820 = 0.00976.

There are three eights left so the chance of a ten-eight combination giving Players 2 and 3 a tie is (2 * 3)/820 = 6/820 = 0.000012

By virtue of having the best hand the probability of Player 3 winning is simply the complement of the sum of the chances of Players 1 or 2 winning and of any tie. That is (820 – 15 – 106 – 8 – 6)/820 = 685/820 = 0.835.

The turn is: 4d

No player's hand improved or even changed much (Player 2 would now use the 4d instead of the 2h in his/her hand). There is no longer a tie possible between any players. Player 1 will now win if the next card is an ace; Player 2 will win if the next card is a nine or a two; Player 3 wins if the next card is anything else.

The probability that Player 1 wins is 3/40 = 0.075.

The probability that Player 2 wins is 5/40 = 0.125

The probability that Player 3 wins is (40 – 3 – 5)/40 = 32/40 = 0.080.

The river card is: 5c

Assuming Player 3 does not fold the winning hand during the betting round before all players show their cards, he/she will be the winner!

These two examples show how mathematics and more specifically, combinatorics, can be applied to games of chance, such as casino games and card games. It can be very difficult to do this counting in real-time as the game is occurring, but knowing the chances for/against you can help you make the decision to stay in the game or fold your hand. Let's now move on to discover what happens to the probabilities of five-card poker hands when jokers and wild cards are used.

Five-Card Stud with Jokers and Wild Cards

Wild Deuces

Included in any standard 52-card deck are two jokers that are optional to any card game. If I now include them as wild cards (i.e., they can be used as any card of the player's choice, including one that has already been used) then there are now 54 total cards. Although they are usually considered as indistinct when playing poker games with them, for the ease of counting and computation I will treat them as distinct. Therefore there are ${}_{54}C_5 = 3{,}162{,}510$ possible hands now, including the addition of a new hand: the five-of-a-kind, which consists of all five cards of the same rank. Along with assuming distinct jokers I also will assume that a player always takes the highest hand possible. This assumption will prove to make the counting more difficult and require more thought for both this case and the case of wild deuces.

Five-of-a-kind (5OAK)

To make this with one joker the hand must be in the form of a natural 4OAK plus the one joker. Since there are 13 ranks possible and one way for the 4OAK to occur in a rank, then coupled with the two choices of jokers there are 13 * 2 = 26 possible 5OAKs with one joker. With two jokers the hand must be a natural 3OAK with the other two jokers. Again there are 13 ranks possible with ${}_4C_3$ possible combinations for 3OAK. Since there is only one two-joker combination then the total number of 5OAKs with two jokers is $13 * {}_4C_3 * 1 = 52$. Since there are no natural 5OAKs then the total number of possible 5OAKs is 26 + 52 = 78.

Straight Flush

There are 40 naturally occurring straight flushes, four of which are of the highest sequence of 10 – J – Q – K – A which make them royal flushes. With one joker the hand must consist of four of the five cards needed to make the straight flush. The one joker can act as any card in the sequence except for the lowest-ranked card. This is because if it acted as the lowest ranked card then it could be counted as the highest ranked card for the next higher ranked straight flush (i.e., if the sequence is X, X+1, X+2, X+3, X+4, then the joker cannot act as the card of rank X because then it could be counted as rank X+5 and give a higher straight flush. This is due to the assumption that the player would always take the best hand.) This pattern holds for the lowest nine sequences (i.e., those that start with A, 2, 3,…, 8, 9). For the highest sequence of 10 – A any card can be replaced by the joker. Thus there are 2(4 * 9 * 4 + 5 * 4) = 328 straight flushes with one joker, 40 of which are royal flushes. For two jokers the same pattern holds: neither of the jokers can act as the lowest ranked card of a sequence except for in the 10 – A sequence. Now there are ${}_4C_2$ ways the jokers can replace any cards in one of the first nine sequences, and ${}_5C_2$ ways for the 10 – A sequence. Thus there are ${}_4C_2 * 9 * 4 + {}_5C_2 * 4 =$ 256 straight flushes with two jokers, 40 of which are royal flushes. The total is then 40 + 328 + 256 = 624 straight flushes, 4 + 40 + 40 = 84 of which are royal flushes.

Four-of-a-kind

For a 4OAK with one joker the hand must be in the form of a natural 3OAK with one joker and another card of a different rank. Note that a hand of this form is automatically a full house, but since the player will take the highest hand possible (which I hypothesize is 4OAK over a full house), I will assume the player takes the four-of-a-kind. There are 13 ranks and ${}_4C_3$ 3OAKs per rank with 48 other cards that are of a different rank. Thus there are $13 * {}_4C_3 * 2 * 48 = 4992$ 4OAKs with one joker. With two jokers the hand must be in the form of a natural pair with the two jokers and another card of a different rank. Thus there are $13 * {}_4C_2 * 1 * 48 = 3744$ 4OAKs with two jokers.

Added to the 624 natural 4OAKs then there are 4992 + 3744 + 624 = 9360 possible fours-of-a-kind.

Full House

Note that it is impossible to make a full house with two jokers as any attempt to do so would result in a hand being either a 3OAK or 4OAK. With one joker the hand must consist of two natural pair with the joker. However there is a matter of "what full of what". Full houses are termed as "Xs full of Ys" where the X is the rank of the 3OAK and the Y is the rank of the pair. Since the player will always be taking the best possible hand a good approach to counting the number of full houses is to start with the highest full house, "Aces full of X". X can be anything between K and 2 in this case (there are 12 A/X combinations) and there $({}_4C_2)^2$ possible pairs between the two ranks. Thus there are 12 * $({}_4C_2)^2$ * 2 = 864 full houses in this case. Going down to "Kings full of X" and using the same argument leads to 11 * $({}_4C_2)^2$ * 2 = 792 possibilities. A pattern emerges all the way down to "Threes full of X" where X can only be a 2. There are no "Twos full of X" full house possibilities due to the assumption of best hand possible. The pattern results in 2 * $({}_4C_2)^2$ * (12 + 11 + 10 + … + 3 + 2 + 1) = 5616 possible full houses with one joker. Along with the 3744 naturally occurring full houses there are 3744 + 5616 = 9360 total full houses possible.

Flush

With one joker a flush can be made with four cards of the same suit plus the joker. There are ${}_{13}C_4$ ways to have four cards of the same suit, 4 suits to choose from, and 2 jokers. Using the Venn Diagram argument (there is an overlap of 328 straight flushes) there are ${}_{13}C_4$ * 4 * 2 – 328 = 5392 possible flushes with one joker. With two jokers there must be three cards of the same suit plus the two jokers. Thus there are ${}_{13}C_3$ * 4 * 1 – 256 = 888 flushes possible with two jokers. Adding in the naturally occurring flushes gives the total number of flushes possible as 5392 + 888 + 5108 = 11,388.

Straight

The argument for counting the number of straights follows similarly from that of counting straight flushes. However, now there are four choices for the card of each rank in the sequence. For one joker, for the first nine sequences (starting at A, 2, 3, … , 8, 9) only four cards can be replaced by the joker. Due to the joker then each of those four ranks actually has five choices. For the highest sequence any card can be replaced, giving five cards with five choices. Thus there are $2(5^4 * 9 + 5^5) – 328$ = 50,922 flushes with one joker (328 straight flushes must be removed). For two jokers a similar argument as for straight flushes also holds here, too. In the first nine sequences the two jokers can replace two cards in ${}_4C_2$ ways with the other three cards having four choices each. The cards in the last sequence can be replaced by the jokers in ${}_5C_2$ ways with four choices for each also. This leaves the count as ${}_4C_2 * 9 * 4^3 + {}_5C_2 * 1 * 4^3 – 256$ = 3840 straights with two jokers. The total possible is then 50922 + 3840 + 10200 = 64,962 straights.

Three-of-a-kind

With one joker the hand must consist of a natural pair and two other cards of different rank from each other and from the pair. There are 13 ranks, ${}_4C_2$ pairs per rank, and two jokers. The last two cards have 48 and 44 choices due to removing the four cards of each rank from the remaining cards. To remove the permutations and obtain only combinations I must divide by 2!. The total number of 3OAKs is then 13 * ${}_4C_2$ * 2 * (48 * 44)/2! = 164,736 with one joker. With two jokers the hand must consist of the two jokers and three other cards that are all unpaired, non-sequential, and unsuited. I must also remove the 3! permutations in favor of combinations. This results in (52 * 48 * 44)/3! * 1 – 3840 – 256 – 888 = 13,320 possible 3OAKs with two jokers. The total number of threes-of-a-kind is then 164736 + 13320 + 54912 = 232,968.

Two Pair

This hand is not possible with either one or two jokers. Any attempts to get such a hand with jokers would result in either one pair, 3OAK, or some other better hand. Thus the total number is the same as the number of naturally occurring two pairs, which is 123,552.

One Pair

One pair is only possible with one joker. The hand must consist of the one joker and the other four cards all being unpaired, non-sequential, and unsuited. The 4! permutations must also be converted to combinations. The number of possible one pair hands with one joker is thus (52 * 48 * 44 * 40)/4! * 2 – 328 – 5392 – 50922 = 309,438. With the natural one pair hands the total is 309438 + 1098240 = 1,407,678.

High Card

This hand is not possible with jokers for obvious reasons. The total is the same as the number of naturally occurring high card hands: 1,302,540.

The total number of hands is 78 + 624 + 9360 + 9360 + 11388 + 64962 + 232968 + 123552 + 1407678 + 1302540 = 3,162,510 = ${}_{54}C_5$.

The following table summarizes the numbers of, and probabilities for, each hand with jokers.

Five-Card Stud with Wild Jokers		
Hand	**Number**	**Probability**
Royal Flush	84	0.00003
Five-of-a-Kind	78	0.00003
Straight Flush	540	0.00017
Four-of-a-Kind	9360	0.00296
Full House	9360	0.00296
Flush	11,388	0.00360
Straight	64,962	0.02054
Three-of-a-Kind	232,968	0.07367
Two Pair	123,552	0.03907
One Pair	1,407,678	0.44511
High Card	1,302,540	0.41187
Total	3,162,510	1.00001*

The most interesting things to note are the relationships between numbers of each hand possible. It turns out that fives-of-a-kind are now more rare than royal flushes! Not only that, but there are an equal number of fours-of-a-kind and full houses! Another inversion is between threes-of-a-kind and two pairs and between one pairs and high card hands. Each of those pairs of numbers is flipped. According to the theory for ranking hands being in terms of the rarity of a given hand then a new ranking for five-card stud with jokers could be:

1) Five-of-a-Kind
2) Straight Flush/Royal Flush
3) Four-of-a-Kind[4]
4) Full House
5) Flush
6) Straight
7) Two Pair
8) Three-of-a-Kind
9) High Card
10) One Pair

This shows how making certain cards wild can cause noticeable fluctuations in the probabilities of a given hand. Now let's see how things change when twos are made wild instead of jokers (with jokers no longer being used).

[4] Since I assumed 4OAKs were higher than full houses to do the counting I must rank them over full houses despite both having the same number of hands. Had I instead considered full houses as the better hand then the 4992 4OAKs with one joker would be part of the 5616 full houses with one joker. But then the total number of hands would no longer sum to ${}_{54}C_5$. Therefore in order to avoid any contradictions I must rank the hands the way I did.

Wild Deuces

I will now derive the probabilities of the given poker hands in five-card stud with all twos being wild. This is commonly referred to as "deuces wild poker." I predict the number of ways of making each hand will change from the case with jokers wild since the wild cards are now members of the original 52 cards. Also there will be a larger number of hands that will not be possible to make. I will point those out as I go. I will hold the same assumptions as before, with a few additions: I will assume the ranks of hands are the same as they were for the natural case (with 5OAKS added as the top hand), and I will assume the twos are "totally wild", meaning they are always counted as a wild card, even if they act as themselves. The latter assumption will help avoid some confusion when counting certain hands. Since the deuces are all distinct then the number of possible hands should sum to ${}_{52}C_5$, or 2,598,960. I will consider all hands for the cases of naturally occurring, with one deuce, with two deuces, with three deuces, and with four deuces.

Five-of-a-Kind

Note that it is not possible to have a 5OAK in twos. That leaves 12 ranks that can serve as 5OAK hands. It is obvious that a 5OAK cannot occur naturally.

With one deuce the hand must consist of a natural 4OAK with one deuce. Thus there are $12 * {}_4C_4 * {}_4C_1 = 48$ 5OAKs with one deuce.

With two deuces the hand must consist of a natural 3OAK with two deuces. The number there is $12 * {}_4C_3 * {}_4C_2 = 288$.

With three deuces the hand must consist of a natural pair with three deuces. The number of those is $12 * {}_4C_2 * {}_4C_3 = 288$.

With four deuces the number is $12 * {}_4C_1 * {}_4C_4 = 48$. There is a debate at this point whether these kinds of hands should be considered as straight flushes or not, and I will discuss that after deriving the number of straight flushes. The grand total for the number of fives-of-a-kind is $2 * 48 + 2 * 288 = 672$.

Straight Flush/Royal Flush

There are enough cases to consider such that I will consider royal flushes separately from straight flushes.

There are four naturally occurring royal flushes. This is trivial.

With one deuce there are 4 suits with ${}_4C_1$ deuces to choose to use and 5 different cards that the deuce can act as. Thus the number is $5 * {}_4C_1 * 4 = 80$.

With two deuces the ${}_4C_2$ possible deuces can now act as ${}_5C_2$ possible cards. Thus the number is ${}_5C_2 * {}_4C_2 * 4 = 240$. The same trend continues for three deuces. The number for those is ${}_5C_3 * {}_4C_3 * 4 = 160$.

There is a problem with royal flushes with four deuces, however. If I were to assume they would count as royal flushes then the number would be ${}_5C_4 * {}_4C_4 * 4 = 20$. However, I will argue that these hands should instead be considered 5OAKs. If not then the numbers of each hands would have to be adjusted. The number of 5OAKs would be 652 because the 20 5OAK hands with four deuces would be counted as royal flushes, and the total number of royal flushes would be 504. This agrees with the theory of ranking of poker hands, but one must remember that royal flushes are not a special type of hand; they are simply straight flushes. If I add in the number of straight flushes (which I will count next) then I will get a much larger number, making 5OAKs more rare, and that will justify giving 5OAKs a higher ranking and allow me to consider the 20 hands in question as 5OAKs. To consider royal flushes as a special hand on their own I would have to also consider Ace-high straights and flushes, and other various hands, as separate, too. The resolution of this debate is that the 20 hands in question are 5OAKs and the total number of royal flushes is 4 + 80 + 240 + 160 = 484.

For straight flushes, note that the 2 – 6 straight flush is not possible since having a deuce would automatically improve such a hand to a 3 – 7 straight flush. Also due to the definition of deuces as "totally wild", I consider the A – 5 straight flush as a wild one. Therefore there are 28 naturally occurring straight flushes (36 without wilds – 4 of the 2-6 variety – 4 of the A-5 variety).

For straight flushes with wilds it is best to break up each situation into three cases: the A – 5 sequence, the 10 – A sequence, and the other 7 sequences in-between. With one deuce the A – 5 straight flush can be made in 16 ways (4 suits * ${}_4C_1$ choices of the deuce, but it must take the place of the two in the sequence); the remaining can be made in 448 ways (4 suits * ${}_4C_1$ choices of the deuce * ${}_4C_1$ cards that the deuce can act as * 7 sequences). Thus the total number of straight flushes with one deuce is 464.

With two deuces the A – 5 straight flush can be made in 72 ways (4 suits * ${}_4C_2$ choices of deuces * 3 ways the ${}_4C_2$ deuces can act as other cards); the remaining can be made in 1008 ways (4 suits * ${}_4C_2$ choices of deuces * ${}_4C_2$ ways they can replace other cards * 7 sequences). The total number of straight flushes with two deuces is thus 1080.

With three deuces the A – 5 straight flush can be made in 48 ways (4 * ${}_4C_3$ * 3 ways the deuces can act); the remaining can be made in 448 ways (4 * ${}_4C_3$ * ${}_4C_3$ * 7). Thus the total number of straight flushes with three deuces is 496. It is not possible to make a straight flush with four deuces as the hand would be counted as a 5OAK in that case. This makes the grand total of straight flushes 28 + 464 + 1080 + 496 = 2068.

Four-of-a-Kind

Note that it is not possible to have a 4OAK hand of twos as any hand attempting that would either be a 5OAK or a 3OAK. Therefore there are 12 ranks that can serve as 4OAK hands. The number of naturally occurring 4OAKs is 12 * ${}_4C_4$ * 44 choices for the last card (52 – 4 deuces – 4 cards of the rank of the 4OAK) = 528.

With one deuce the number of 4OAKs is 12 ranks * ${}_4C_3$ 3OAKs per rank * ${}_4C_1$ deuce * 44 choices for the last card = 8448.

With two deuces the number of 4OAKs is 12 * ${}_4C_2$ pairs per rank * ${}_4C_2$ deuces * 44 choices for the last card = 19,008.

With three deuces I find it easier to count each rank separately using the following method. Starting at the top: For four Aces the number is ${}_4C_3$ deuces * ${}_4C_1$ aces * (number of cards that don't improve the hand, which in this case are any ace, deuce, or a suited king, queen, jack, ten, three, four, or five…i.e., 52 – 4 – 4 – 7 = 37) = 592. For kings the first two factors remain the same, and the last card cannot be an ace, king, deuce, or a suited queen, jack, ten, or nine. This leaves 36 possible last cards. The number for four

Kings is thus 576. This pattern continues through the ranks down to fours. The number of last cards possible for the other ranks, starting with jacks, are: 32, 28, 24, 20, 16, 12, 9, 6, and 3. Thus the number of 4OAK hands possible with three deuces is ${}_4C_3 * {}_4C_1 * (37 + 36 + 32 + 28 + 24 + 20 + 16 + 12 + 9 + 6 + 3) = 3568$. Since it is not possible to make a 4OAK with four deuces, the grand total for the number of 4OAKs is 3568 + 8448 + 19008 + 3568 = 31,552.

Full House

Note that a full house consisting of "deuces full of X" or "X full of deuces" (where X is any other rank) is not possible. For naturally occurring full houses there are 12 * 11 ranks possible for the 3OAK and pair and ${}_4C_3 * {}_4C_2$ possible pairs and 3OAKs for each rank. Thus the number of natural full houses is $12 * 11 * {}_4C_3 * {}_4C_2 = 3168$.

With one deuce the hand must consist of a natural two pair plus the one deuce. Just as in the argument for full houses with jokers, the issue of "Xs full of Ys" must be acknowledged. Thus thinking of the number of hands by rank helps. For example, with "Aces full of Xs", there are ${}_4C_2$ aces * ${}_4C_1$ deuces * 11 other ranks that can be used as the other natural pair (13 ranks – aces – deuces) * ${}_4C_2$ pairs in those ranks = 1584. For "Kings full of Xs" everything remains the same except for the number of other ranks that can be included as the other pair, which reduces to 10. The pattern continues down to "Fours full of Xs" in which only one other rank, threes, is possible for the other natural pair. A full house of "Threes over X" is not possible. The calculation for the total number of full houses with one deuce is thus $({}_4C_2)^2 * {}_4C_1 * (11 + 10 + 9 \ldots + 3 + 2 + 1) = 9504$.

It is not possible to get a full house with two, three, or four deuces, as a 4OAK or better would be the result of any attempt at the hand. Thus the grand total for the number of full houses possible with deuces wild is 3168 + 9504 = 12,672.

Flush

The definition of deuces as "totally wild" helps in counting the number of flushes. For the naturally occurring flushes there are 4 suits possible and ${}_{12}C_5$ combinations of five cards of the same suit of each suit (the deuces are excluded since having them in the hand would make it a wild hand, not a natural one). Removing the natural straight flushes and royal flushes from this count gives a total of $4 * {}_{12}C_5 – 28 – 4 = 3136$ natural flushes.

With one deuce there are 4 suits, ${}_4C_1$ deuces to use, and ${}_{12}C_4$ possible combinations of four cards of the same suit (deuce excluded to prevent overcounting). Removing the straight and royal flushes gives a total of $4 * {}_4C_1 * {}_{12}C_5 – 80 – 464 = 7376$ flushes with one deuce.

With two deuces the same pattern holds as for one deuce. The number of hands is $4 * {}_4C_2 * {}_{12}C_3 – 240 – 1080 = 3960$.

A flush is not possible with three or four deuces. Having three or four deuces would give an automatic 4OAK or 5OAK. Thus the grand total for the number of flushes is 3136 + 7376 + 3960 = 14,472.

Straight

Note that a straight of 2 – 6 is not possible with deuces wild. Like the straight flush argument there are three cases to consider: the A – 5 sequence, the A – 10 sequence, and the 7 other possible sequences between those two. For the naturally occurring case, the A – 5 sequence would be considered wild since deuces are totally wild. For the other sequences there are 4 cards for each of the 5 ranks in the sequence. After removing the royal and straight flushes the number is $4^5 * 8 - 4 - 28 = 8160$.

With one deuce in the first case the deuce must act as itself. Therefore each rank in the sequence has 4 cards to choose from. The total number there is just $4^5 = 1024$ (higher-ranked hands that are included in this count will be subtracted at the end of all of the cases). For the case of A – 10 the ${}_4C_1$ deuces can act as ${}_5C_1$ ranks with the other 4 ranks having 4 choices each. The number there is ${}_4C_1 * {}_5C_1 * 4^4 = 5120$. For the 7 in-between sequences the ${}_4C_1$ deuces can act as ${}_4C_1$ cards (not the lowest rank) with the other 4 ranks having 4 choices each. That makes the total as $7 * ({}_4C_1)^2 * 4^4 = 28{,}672$. The total number of straights with one deuce, after removing higher hands, is 1024 + 5120 + 28672 – 80 – 464 = 34,272.

With two deuces in the first case there are 3 ways the ${}_4C_2$ deuces can act as other ranks. Each of the other 3 ranks has 4 choices. Thus the number is $3 * {}_4C_2 * 4^3 = 1152$. For the A – 10 case the ${}_4C_2$ deuces can act as ${}_5C_2$ ranks with the 3 other cards again having 4 choices. The number there is ${}_5C_2 * {}_4C_2 * 4^3 = 3840$. For the middle 7 sequences the ${}_4C_2$ deuces can go in ${}_4C_2$ places with 4^3 choices for the other cards for a total of $7 * ({}_4C_2)^2 * 4^3 = 16{,}128$. After removing the higher hands the total number of straights with two deuces is 1152 + 3840 + 16128 – 240 – 1080 = 19,800.

It is not possible to get a straight with three or four deuces for the same reasons as for flushes. Thus the grand total for the number of straights is 8160 + 34272 + 19800 = 62,232.

Three-of-a-Kind

It is trivial that at 3OAK in deuces is not possible. For naturally occurring 3OAKs there are 12 ranks possible, ${}_4C_3$ possible 3OAKs per rank, and (44 * 40)/2 possible combinations of the last two cards that don't give a better hand (same argument as from previous sections). Thus the number of natural 3OAKs is $12 * {}_4C_3 * (44 * 40)/2 = 42{,}240$.

With one deuce the hand must include a natural pair with the deuce and two other cards that don't improve the hand. Similar to the natural case (the paragraph above), the number is $12 * {}_4C_2 * {}_4C_1 * (44 * 40)/2 = 253{,}440$.

With two deuces the hand must be the two deuces with three unpaired, non-sequential, unsuited cards. Taking the same approach as with the other 3OAK cases, there are 12 ranks, ${}_4C_2$ deuces, 4 cards per rank, and (44 * 40)/2 other cards that would be unpaired. However, in counting the hands this way I have counted each hand 3 times. Thus the formula must be divided by 3. Then the number of higher hands must be removed. The number of 3OAKs possible with two deuces is thus $(12 * 4 * {}_4C_2 * (44 * 40)/2)/3 - 240 - 1080 - 3960 - 19800 = 59{,}400$.

It is not possible to get a 3OAK with three or four deuces for obvious reasons. The grand total for the number of threes-of-a-kind with deuces is 42240 + 253440 + 59400 = 355,080.

Two Pair

This is a simple count since it is not possible to make a two pair hand with any number of deuces. It is only possible to get two pair naturally. There are ${}_{12}C_2$ possible combinations of the ranks of the two pair, ${}_4C_2$ pairs per rank, and 40 other cards that don't improve the hand. Thus the count is ${}_{12}C_2 * ({}_4C_2)^2 * 40 = 95{,}040$.

One Pair

A one pair hand cannot be made in deuces for obvious reasons. For the naturally occurring case there are 12 ranks, ${}_4C_2$ pairs per rank, and (44 * 40 * 36)/6 possible combinations of the other three cards. Thus the number of natural one pair hands is $12 * {}_4C_2 * (44 * 40 * 36)/6 = 760{,}320$.

With one deuce the hand must consist of the deuce and four other unpaired, non-sequential, unsuited cards. It is not possible to get a pair of threes, fours, or fives. For example, try making a hand that is one pair of Fives with a deuce. Such a hand could be 2, 5, 4, 3, …. As you can see even trying to make fives the highest rank would cause a 3OAK or straight or better to result. Therefore doing a rank-by-rank count will suffice. For aces there are ${}_4C_1$ aces to choose from, ${}_4C_1$ deuces, and (44 * 40 * 36)/6 choices for the other cards. For kings the pattern remains the same, but the choices for the other cards is reduced to (40 * 36 * 32)/6. Following the pattern down to sixes, and removing higher hands that are included in the count, results in a formula for the number of one pair hands with one deuce as $({}_4C_1)^2 * [(44 * 40 * 36)/6 + (40 * 36 * 32)/6 + (36 * 32 * 28)/6 + \ldots + (16 * 12 * 8)/6 + (12 * 8 * 4)/6] – 80 – 464 – 7376 – 34272 = 464{,}688$.

Since it is not possible to get a one pair hand with two, three, or four deuces, then the grand total for the number of one pair hands with wild deuces is 760320 + 464688 = 1,225,008.

High Card

Obviously this hand is only possible naturally. I will count the hands manually instead of by elimination of all the other hands from the total number of possible hands (i.e., ${}_{52}C_5$). The cards need to be all non-deuce, non-sequential, unpaired, and unsuited. After removing any higher hands that still could occur (i.e., flushes, straights, straight flushes, and royal flushes) the number of high card hands is given by (48 * 44 * 40 * 36 * 32)/120 – 4 – 28 – 3136 – 8160 = 799,680.

The grand total for the number of hands, which also serves as a check on the counting, is 672 + 484 + 2068 + 31552 + 12672 + 14472 + 62232 + 355080 + 95040 + 1225008 + 799680 = 2,598,960 = ${}_{52}C_5$. The table below shows the number of hands possible, broken down by how many deuces were used to make it.

Five-Card Stud with Deuces Wild							
Hand	**Number of Deuces Used to Make the Hand**					**Number**	**Probability**
	Zero	**One**	**Two**	**Three**	**Four**		
5OAK	0	48	288	288	48	672	0.00026
R. Flush	4	80	240	160	0	484	0.00019[5]
S. Flush	28	464	1080	496	0	2068	0.00080
4OAK	528	8448	19,008	3568	0	31,552	0.01214
Full House	3168	9504	0	0	0	12,672	0.00488
Flush	3136	7376	3960	0	0	14,472	0.00557
Straight	8160	34,272	19,800	0	0	62,232	0.02394
3OAK	42,240	253,440	59,400	0	0	355,080	0.13662
Two Pair	95,040	0	0	0	0	95,040	0.03657
One Pair	760,320	464,688	0	0	0	1,225,008	0.47135
High Card	799,680	0	0	0	0	799,680	0.30769
Total	1,712,304	778,320	103,776	4512	48	2,598,960	1.000014*

In comparing this table to the tables for the other poker situations there are some similarities and some dissimilarities. Again fives-of-a-kind are the most rare with royal and straight flushes close behind. But the big difference that has not previously been seen is the number of fours-of-a-kind. They are far more common than other closely-ranked hands. Both full houses (which for the jokers case had the same number of hands as 4OAKs) and flushes are now more rare than fours-of-a-kind. From there on down the pattern remains similar to what we saw in the case of jokers with threes-of-a-kind being more common than two pair and one pair being more common than high card hands. According to the theory of ranking poker hands, the following list should be the order of hands for deuces wild five-card stud poker:

1) Five-of-a-kind
2) Royal/Straight Flush
3) Full House
4) Flush
5) Four-of-a-Kind
6) Straight
7) Two Pair
8) Three-of-a-Kind
9) High Card
10) One Pair

This is another example of how making certain cards wild can change the probability of getting a given poker hand. Comparing the previous two lists of hands to the first one in the paper (for natural poker) you can see a lot of rearranging of the hands. Although I did not simulate any Texas Hold em hands with wild cards, I predict that if I did the probabilities would change, perhaps significantly (e.g., if deuces were wild then Player 2 would have won the second simulated game with a three-of-a-kind in nines). In games I have played with wild cards the rankings of the hands was not changed, perhaps

[5] With the numbers for royal and straight flushes combined the result is 2552 and 0.00098

because of lack of knowledge of the probabilities or lack of knowledge of poker theory. The way the games with wild cards would be played would probably change, however. For example people would not play a hand such as a three-of-a-kind knowing that even two pair would beat it. More studies could be done to discuss this, but that is not the focus of my paper. My focus was to discover the patterns in the numbers and probabilities of the poker hands for the cases of natural, jokers wild, and deuces wild poker, and to simulate Texas Hold em games to see how the probability of a player winning changed throughout the game. Having accomplished that I leave the reader to think about the patterns of numbers and probabilities of the hands in other variations of poker, such as wild sevens, draw poker, and Omaha.