9.4 Multiplying and Dividing Radicals

9.4 OBJECTIVES

1. Multiply and divide expressions involving numeric radicals
2. Multiply and divide expressions involving algebraic radicals

In Section 9.2 we stated the first property for radicals:

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ when a and b are any positive real numbers

That property has been used to simplify radical expressions up to this point. Suppose now that we want to find a product, such as $\sqrt{3} \cdot \sqrt{5}$.

We can use our first radical rule in the opposite manner.

NOTE The product of square roots is equal to the square root of the product of the radicands.

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

so

$$\sqrt{3} \cdot \sqrt{5} = \sqrt{3 \cdot 5} = \sqrt{15}$$

We may have to simplify after multiplying, as Example 1 illustrates.

Simplifying Radical Expressions

Multiply then simplify each expression.

(a) $\sqrt{5} \cdot \sqrt{10} = \sqrt{5 \cdot 10} = \sqrt{50}$

$$= \sqrt{25 \cdot 2} = 5\sqrt{2}$$

(b) $\sqrt{12} \cdot \sqrt{6} = \sqrt{12 \cdot 6} = \sqrt{72}$

$$= \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}$$

An alternative approach would be to simplify $\sqrt{12}$ first.

$$\sqrt{12} \cdot \sqrt{6} = 2\sqrt{3}\sqrt{6} = 2\sqrt{18}$$

$$= 2\sqrt{9 \cdot 2} = 2\sqrt{9}\sqrt{2}$$

$$= 2 \cdot 3\sqrt{2} = 6\sqrt{2}$$

(c) $\sqrt{10x} \cdot \sqrt{2x} = \sqrt{20x^2} = \sqrt{4x^2 \cdot 5}$

$$= \sqrt{4x^2} \cdot \sqrt{5} = 2x\sqrt{5}$$

CHECK YOURSELF 1

Simplify.

(a) $\sqrt{3} \cdot \sqrt{6}$ **(b)** $\sqrt{3} \cdot \sqrt{18}$ **(c)** $\sqrt{8a} \cdot \sqrt{3a}$

If coefficients are involved in a product, we can use the commutative and associative properties to change the order and grouping of the factors. This is illustrated in Example 2.

Example 2

Multiplying Radical Expressions

Multiply.

$$(2\sqrt{5})(3\sqrt{6}) = (2 \cdot 3)(\sqrt{5} \cdot \sqrt{6})$$
$$= 6\sqrt{5 \cdot 6}$$
$$= 6\sqrt{30}$$

NOTE In practice, it is not necessary to show the intermediate steps.

CHECK YOURSELF 2

Multiply $(3\sqrt{7})(5\sqrt{3})$.

The distributive property can also be applied in multiplying radical expressions. Consider the following.

Multiplying Radical Expressions

Multiply.

(a) $\sqrt{3}(\sqrt{2} + \sqrt{3})$

$= \sqrt{3} \cdot \sqrt{2} + \sqrt{3} \cdot \sqrt{3}$ The distributive property

$= \sqrt{6} + 3$ Multiply the radicals.

(b) $\sqrt{5}(2\sqrt{6} + 3\sqrt{3})$

$= \sqrt{5} \cdot 2\sqrt{6} + \sqrt{5} \cdot 3\sqrt{3}$ The distributive property

$= 2 \cdot \sqrt{5} \cdot \sqrt{6} + 3 \cdot \sqrt{5} \cdot \sqrt{3}$ The commutative property

$= 2\sqrt{30} + 3\sqrt{15}$

CHECK YOURSELF 3

Multiply.

(a) $\sqrt{5}(\sqrt{6} + \sqrt{5})$ **(b)** $\sqrt{3}(2\sqrt{5} + 3\sqrt{2})$

The FOIL pattern we used for multiplying binomials in Section 3.4 can also be applied in multiplying radical expressions. This is shown in Example 4.

Example 4

Multiplying Radical Expressions

Multiply.

(a) $(\sqrt{3} + 2)(\sqrt{3} + 5)$

$= \sqrt{3} \cdot \sqrt{3} + 5\sqrt{3} + 2\sqrt{3} + 2 \cdot 5$

$= 3 + 5\sqrt{3} + 2\sqrt{3} + 10$ Combine like terms.

$= 13 + 7\sqrt{3}$

CAUTION

Be Careful! This result *cannot* be further simplified: 13 and $7\sqrt{3}$ are *not* like terms.

NOTE You can use the pattern $(a + b)(a - b) = a^2 - b^2$, where $a = \sqrt{7}$ and $b = 2$, for the same result. $\sqrt{7} + 2$ and $\sqrt{7} - 2$ are called **conjugates** of each other. Note that their product is the rational number 3. The product of conjugates will *always be rational.*

(b) $(\sqrt{7} + 2)(\sqrt{7} - 2) = \sqrt{7} \cdot \sqrt{7} - 2\sqrt{7} + 2\sqrt{7} - 4$

$= 7 - 4 = 3$

(c) $(\sqrt{3} + 5)^2 = (\sqrt{3} + 5)(\sqrt{3} + 5)$

$= \sqrt{3} \cdot \sqrt{3} + 5\sqrt{3} + 5\sqrt{3} + 5 \cdot 5$

$= 3 + 5\sqrt{3} + 5\sqrt{3} + 25$

$= 28 + 10\sqrt{3}$

CHECK YOURSELF 4

Multiply.

(a) $(\sqrt{5} + 3)(\sqrt{5} - 2)$ **(b)** $(\sqrt{3} + 4)(\sqrt{3} - 4)$ **(c)** $(\sqrt{2} - 3)^2$

We can also use our second property for radicals in the opposite manner.

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

NOTE The quotient of square roots is equal to the square root of the quotient of the radicands.

One use of this property to divide radical expressions is illustrated in Example 5.

Simplifying Radical Expressions

Simplify.

NOTE The clue to recognizing when to use this approach is in noting that 48 is divisible by 3.

(a) $\dfrac{\sqrt{48}}{\sqrt{3}} = \sqrt{\dfrac{48}{3}} = \sqrt{16} = 4$

(b) $\dfrac{\sqrt{200}}{\sqrt{2}} = \sqrt{\dfrac{200}{2}} = \sqrt{100} = 10$

(c) $\dfrac{\sqrt{125x^2}}{\sqrt{5}} = \sqrt{\dfrac{125x^2}{5}} = \sqrt{25x^2} = 5x$

There is one final quotient form that you may encounter in simplifying expressions, and it will be extremely important in our work with quadratic equations in the next chapter. This form is shown in Example 6.

CHECK YOURSELF 5

Simplify.

(a) $\dfrac{\sqrt{75}}{\sqrt{3}}$ **(b)** $\dfrac{\sqrt{81s^2}}{\sqrt{9}}$

Example 6

Simplifying Radical Expressions

Simplify the expression

$$\frac{3+\sqrt{72}}{3}$$

First, we must simplify the radical in the numerator.

$$\frac{3+\sqrt{72}}{3} = \frac{3+\sqrt{36\cdot 2}}{3}$$

Use Property 1 to simplify $\sqrt{72}$.

$$= \frac{3+\sqrt{36}\cdot\sqrt{2}}{3} = \frac{3+6\sqrt{2}}{3}$$

$$= \frac{3(1+2\sqrt{2})}{3} = 1+2\sqrt{2}$$

Factor the numerator—then divide by the *common* factor 3.

CAUTION

Be Careful! Students are sometimes tempted to write

$$\frac{\cancel{3}+6\sqrt{2}}{\cancel{3}} = 1+6\sqrt{2}$$

This is *not* correct. We must divide *both terms* of the numerator by the common factor.

CHECK YOURSELF 6

Simplify $\dfrac{15+\sqrt{75}}{5}$.

CHECK YOURSELF ANSWERS

1. **(a)** $3\sqrt{2}$; **(b)** $3\sqrt{6}$; **(c)** $2a\sqrt{6}$ **2.** $15\sqrt{21}$ **3.** **(a)** $\sqrt{30}+5$; **(b)** $2\sqrt{15}+3\sqrt{6}$ **4.** **(a)** $-1+\sqrt{5}$; **(b)** -13; **(c)** $11-6\sqrt{2}$ **5.** **(a)** 5; **(b)** $3s$ **6.** $3+\sqrt{3}$

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Section __________ Date __________

9.4 Exercises

Perform the indicated multiplication. Then simplify each radical expression.

1. $\sqrt{7} \cdot \sqrt{5}$

2. $\sqrt{3} \cdot \sqrt{7}$

3. $\sqrt{5} \cdot \sqrt{11}$

4. $\sqrt{13} \cdot \sqrt{5}$

5. $\sqrt{3} \cdot \sqrt{10m}$

6. $\sqrt{7a} \cdot \sqrt{13}$

7. $\sqrt{2x} \cdot \sqrt{15}$

8. $\sqrt{17} \cdot \sqrt{2b}$

9. $\sqrt{3} \cdot \sqrt{7} \cdot \sqrt{2}$

10. $\sqrt{5} \cdot \sqrt{7} \cdot \sqrt{3}$

11. $\sqrt{3} \cdot \sqrt{12}$

12. $\sqrt{7} \cdot \sqrt{7}$

13. $\sqrt{10} \cdot \sqrt{10}$

14. $\sqrt{5} \cdot \sqrt{15}$

15. $\sqrt{18} \cdot \sqrt{6}$

16. $\sqrt{8} \cdot \sqrt{10}$

17. $\sqrt{2x} \cdot \sqrt{6x}$

18. $\sqrt{3a} \cdot \sqrt{15a}$

19. $2\sqrt{3} \cdot \sqrt{7}$

20. $3\sqrt{2} \cdot \sqrt{5}$

21. $(3\sqrt{3})(5\sqrt{7})$

22. $(2\sqrt{5})(3\sqrt{11})$

ANSWERS

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23. $(3\sqrt{5})(2\sqrt{10})$

24. $(4\sqrt{3})(3\sqrt{6})$

25. $\sqrt{5}(\sqrt{2} + \sqrt{5})$

26. $\sqrt{3}(\sqrt{5} - \sqrt{3})$

27. $\sqrt{3}(2\sqrt{5} - 3\sqrt{3})$

28. $\sqrt{7}(2\sqrt{3} + 3\sqrt{7})$

29. $(\sqrt{3} + 5)(\sqrt{3} + 3)$

30. $(\sqrt{5} - 2)(\sqrt{5} - 1)$

31. $(\sqrt{5} - 1)(\sqrt{5} + 3)$

32. $(\sqrt{2} + 3)(\sqrt{2} - 7)$

33. $(\sqrt{5} - 2)(\sqrt{5} + 2)$

34. $(\sqrt{7} + 5)(\sqrt{7} - 5)$

35. $(\sqrt{10} + 5)(\sqrt{10} - 5)$

36. $(\sqrt{11} - 3)(\sqrt{11} + 3)$

37. $(\sqrt{x} + 3)(\sqrt{x} - 3)$

38. $(\sqrt{a} - 4)(\sqrt{a} + 4)$

39. $(\sqrt{3} + 2)^2$

40. $(\sqrt{5} - 3)^2$

41. $(\sqrt{y} - 5)^2$

42. $(\sqrt{x} + 4)^2$

Perform the indicated division. Rationalize the denominator if necessary. Then simplify each radical expression.

43. $\dfrac{\sqrt{98}}{\sqrt{2}}$

44. $\dfrac{\sqrt{108}}{\sqrt{3}}$

45. $\dfrac{\sqrt{72a^2}}{\sqrt{2}}$

46. $\dfrac{\sqrt{48m^2}}{\sqrt{3}}$

47. $\dfrac{4 + \sqrt{48}}{4}$

48. $\dfrac{12 + \sqrt{108}}{6}$

49. $\dfrac{5 + \sqrt{175}}{5}$

50. $\dfrac{18 + \sqrt{567}}{9}$

51. $\dfrac{-8 - \sqrt{512}}{4}$

52. $\dfrac{-9 - \sqrt{108}}{3}$

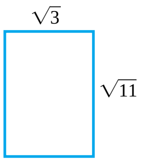
53. $\dfrac{6 + \sqrt{18}}{3}$

53. $\dfrac{6 - \sqrt{20}}{2}$

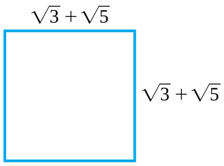
55. $\dfrac{15 - \sqrt{75}}{5}$

56. $\dfrac{8 + \sqrt{48}}{4}$

57. **Area of a rectangle.** Find the area of the rectangle shown in the figure.



58. **Area of a rectangle.** Find the area of the rectangle shown in the figure.



59. Complete this statement: "$\sqrt{2} \cdot \sqrt{5} = \sqrt{10}$ because . . . ."

60. Explain why $2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$ but $7\sqrt{3} + 3\sqrt{5} \neq 10\sqrt{8}$.

61. When you look out over an unobstructed landscape or seascape, the distance to the visible horizon depends on your height above the ground. The equation

$$d = \sqrt{\frac{3}{2}h}$$

is a good estimate of this, in which d = distance to horizon in miles and h = height of viewer above the ground. Work with a partner to make a chart of distances to the horizon given different elevations. Use the actual heights of tall buildings or

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ANSWERS

a. ______

b. ______

c. ______

d. ______

e. ______

f. ______

prominent landmarks in your area. The local library should have a list of these. Be sure to consider the view to the horizon you get when flying in a plane. What would your elevation have to be to see from one side of your city or town to the other? From one side of your state or county to the other?

Getting Ready for Section 9.5 [Section 9.1]

Evaluate the following. Round your answer to the nearest thousandth.

(a) $\sqrt{16}$ (b) $\sqrt{49}$

(c) $\sqrt{121}$ (d) $\sqrt{12}$

(e) $\sqrt{27}$ (f) $\sqrt{98}$

Answers

1. $\sqrt{35}$ **3.** $\sqrt{55}$ **5.** $\sqrt{30m}$ **7.** $\sqrt{30x}$ **9.** $\sqrt{42}$ **11.** 6 **13.** 10 **15.** $6\sqrt{3}$ **17.** $2x\sqrt{3}$ **19.** $2\sqrt{21}$ **21.** $15\sqrt{21}$ **23.** $30\sqrt{2}$ **25.** $\sqrt{10} + 5$ **27.** $2\sqrt{15} - 9$ **29.** $18 + 8\sqrt{3}$ **31.** $2 + 2\sqrt{5}$ **33.** 1 **35.** -15 **37.** $x - 9$ **39.** $7 + 4\sqrt{3}$ **41.** $y - 10\sqrt{y} + 25$ **43.** 7 **45.** $6a$ **47.** $1 + \sqrt{3}$ **49.** $1 + \sqrt{7}$ **51.** $-2 - 4\sqrt{2}$ **53.** $2 + \sqrt{2}$ **55.** $3 - \sqrt{3}$ **57.** $\sqrt{33}$ **59.** **61.** **a.** 4 **b.** 7 **c.** 11 **d.** 3.464 **e.** 5.196 **f.** 9.899