6.6

Finding Rational Zeros

What *you should learn*

GOAL 1 Find the rational zeros of a polynomial function.

GOAL 2 Use polynomial equations to solve **real-life** problems, such as finding the dimensions of a monument in **Ex. 60**.

Why *you should learn it*

▼ To model **real-life** quantities, such as the volume of a representation of the Louvre pyramid in **Example 3**.

REAL LIFE

GOAL 1 USING THE RATIONAL ZERO THEOREM

The polynomial function

$$f(x) = 64x^3 + 120x^2 - 34x - 105$$

has $-\frac{3}{2}$, $-\frac{5}{4}$, and $\frac{7}{8}$ as its zeros. Notice that the numerators of these zeros (−3, −5, and 7) are factors of the constant term, −105. Also notice that the denominators (2, 4, and 8) are factors of the leading coefficient, 64. These observations are generalized by the *rational zero theorem.*

THE RATIONAL ZERO THEOREM

If $f(x) = a_nx^n + \cdots + a_1x + a_0$ has *integer* coefficients, then every rational zero of f has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

EXAMPLE 1 *Using the Rational Zero Theorem*

Find the rational zeros of $f(x) = x^3 + 2x^2 - 11x - 12$.

SOLUTION

List the possible rational zeros. The leading coefficient is 1 and the constant term is −12. So, the possible rational zeros are:

$$x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}$$

Test these zeros using synthetic division.

Test $x = 1$:

1	1	2	−11	−12
		1	3	−8
	1	3	−8	−20

Test $x = -1$:

−1	1	2	−11	−12
		−1	−1	12
	1	1	−12	0

Since −1 is a zero of f, you can write the following:

$$f(x) = (x + 1)(x^2 + x - 12)$$

Factor the trinomial and use the factor theorem.

$$f(x) = (x + 1)(x^2 + x - 12) = (x + 1)(x - 3)(x + 4)$$

▶ The zeros of f are −1, 3, and −4.

In Example 1, the leading coefficient is 1. When the leading coefficient is not 1, the list of possible rational zeros can increase dramatically. In such cases the search can be shortened by sketching the function's graph—either by hand or by using a graphing calculator.

EXAMPLE 2 *Using the Rational Zero Theorem*

Find all real zeros of $f(x) = 10x^4 - 3x^3 - 29x^2 + 5x + 12$.

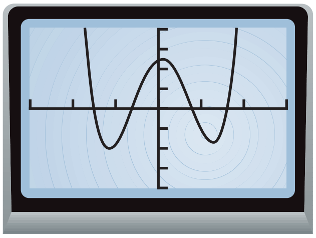
SOLUTION

List the possible rational zeros of f: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1},$

$\pm\frac{12}{1}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}, \pm\frac{12}{10}.$

Choose values to check.

With so many possibilities, it is worth your time to sketch the graph of the function. From the graph, it appears that some reasonable choices are $x = -\frac{3}{2}$, $x = -\frac{3}{5}$, $x = \frac{4}{5}$, and $x = \frac{3}{2}$.



Check the chosen values using synthetic division.

$-\frac{3}{2}$	10	−3	−29	5	12
		−15	27	3	−12
	10	−18	−2	8	0

← $-\frac{3}{2}$ **is a zero.**

Factor out a binomial using the result of the synthetic division.

$$f(x) = \left(x + \frac{3}{2}\right)(10x^3 - 18x^2 - 2x + 8) \quad \text{Rewrite as a product of two factors.}$$

$$= \left(x + \frac{3}{2}\right)(2)(5x^3 - 9x^2 - x + 4) \quad \text{Factor 2 out of the second factor.}$$

$$= (2x + 3)(5x^3 - 9x^2 - x + 4) \quad \text{Multiply the first factor by 2.}$$

Repeat the steps above for $g(x) = 5x^3 - 9x^2 - x + 4$.

Any zero of g will also be a zero of f. The possible *rational* zeros of g are $x = \pm 1, \pm 2, \pm 4, \pm\frac{1}{5}, \pm\frac{2}{5}$, and $\pm\frac{4}{5}$. The graph of f shows that $\frac{4}{5}$ may be a zero.

$\frac{4}{5}$	5	−9	−1	4
		4	−4	−4
	5	−5	−5	0

← $\frac{4}{5}$ **is a zero.**

So $f(x) = (2x + 3)\left(x - \frac{4}{5}\right)(5x^2 - 5x - 5) = (2x + 3)(5x - 4)(x^2 - x - 1)$.

Find the remaining zeros of f by using the quadratic formula to solve $x^2 - x - 1 = 0$.

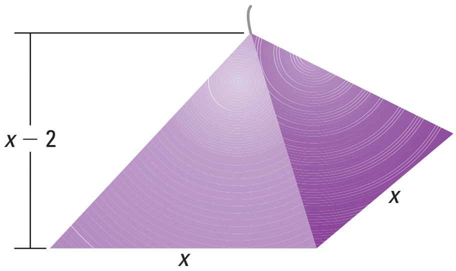
▶ The real zeros of f are $-\frac{3}{2}, \frac{4}{5}, \frac{1 + \sqrt{5}}{2}$, and $\frac{1 - \sqrt{5}}{2}$.

GOAL 2 SOLVING POLYNOMIAL EQUATIONS IN REAL LIFE

REAL LIFE
Crafts

EXAMPLE 3 *Writing and Using a Polynomial Model*

You are designing a candle-making kit. Each kit will contain 25 cubic inches of candle wax and a mold for making a model of the pyramid-shaped building at the Louvre Museum in Paris, France. You want the height of the candle to be 2 inches less than the length of each side of the candle's square base. What should the dimensions of your candle mold be?



SOLUTION

The volume is $V = \frac{1}{3}Bh$ where B is the area of the base and h is the height.

PROBLEM SOLVING STRATEGY

VERBAL MODEL Volume $= \frac{1}{3} \cdot$ Area of base $\cdot$ Height

LABELS

Volume = **25**		(cubic inches)
Side of square base = x		(inches)
Area of base = x^2		(square inches)
Height = $x - 2$		(inches)

ALGEBRAIC MODEL

$25 = \frac{1}{3}x^2(x - 2)$ — **Write algebraic model.**

$75 = x^3 - 2x^2$ — **Multiply each side by 3 and simplify.**

$0 = x^3 - 2x^2 - 75$ — **Subtract 75 from each side.**

The possible rational solutions are $x = \pm\frac{1}{1}, \pm\frac{3}{1}, \pm\frac{5}{1}, \pm\frac{15}{1}, \pm\frac{25}{1}, \pm\frac{75}{1}$.

Use the possible solutions. Note that in this case, it makes sense to test only positive x-values.

1	1	−2	0	−75
		1	−1	−1
	1	−1	−1	−76

3	1	−2	0	−75
		3	3	9
	1	1	3	−66

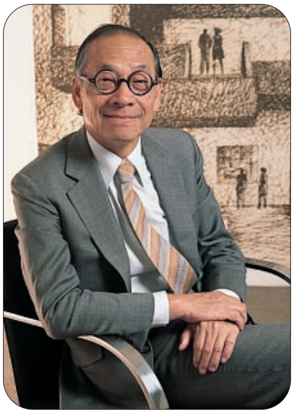
5	1	−2	0	−75
		5	15	75
	1	3	15	0

← **5 is a solution.**

So $x = 5$ is a solution. The other two solutions, which satisfy $x^2 + 3x + 15 = 0$, are $x = \frac{-3 \pm i\sqrt{51}}{2}$ and can be discarded because they are imaginary.

▶ The base of the candle mold should be 5 inches by 5 inches. The height of the mold should be $5 - 2 = 3$ inches.

FOCUS ON PEOPLE



REAL LIFE **I.M. PEI** designed the pyramid at the Louvre. His geometric architecture can be seen in Boston, New York, Dallas, Los Angeles, Taiwan, Beijing, and Singapore.

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement of the rational zero theorem: If a polynomial function has integer coefficients, then every rational zero of the function has the form $\frac{p}{q}$, where p is a factor of the __?__ and q is a factor of the __?__.

Concept Check ✓

2. For each polynomial function, decide whether you can use the rational zero theorem to find its zeros. Explain why or why not.

a. $f(x) = 6x^2 - 8x + 4$ **b.** $f(x) = 0.3x^2 + 2x + 4.5$ **c.** $f(x) = \frac{1}{4}x^2 - x + \frac{7}{8}$

3. Describe a method you can use to shorten the list of possible rational zeros when using the rational zero theorem.

Skill Check ✓

List the possible rational zeros of f using the rational zero theorem.

4. $f(x) = x^3 + 14x^2 + 41x - 56$

5. $f(x) = x^3 - 17x^2 + 54x + 72$

6. $f(x) = 2x^3 + 7x^2 - 7x + 30$

7. $f(x) = 5x^4 + 12x^3 - 16x^2 + 10$

Find all the real zeros of the function.

8. $f(x) = x^3 - 3x^2 - 6x + 8$

9. $f(x) = x^3 + 4x^2 - x - 4$

10. $f(x) = 2x^3 - 5x^2 - 2x + 5$

11. $f(x) = 2x^3 - x^2 - 15x + 18$

12. $f(x) = x^3 + 4x^2 + x - 6$

13. $f(x) = x^3 + 5x^2 - x - 5$

14. **CRAFTS** Suppose you have 18 cubic inches of wax and you want to make a candle in the shape of a pyramid with a square base. If you want the height of the candle to be 3 inches greater than the length of each side of the base, what should the dimensions of the candle be?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 948.

LISTING RATIONAL ZEROS **List the possible rational zeros of f using the rational zero theorem.**

15. $f(x) = x^4 + 2x^2 - 24$

16. $f(x) = 2x^3 + 5x^2 - 6x - 1$

17. $f(x) = 2x^5 + x^2 + 16$

18. $f(x) = 2x^3 + 9x^2 - 53x - 60$

19. $f(x) = 6x^4 - 3x^3 + x + 10$

20. $f(x) = 4x^3 + 5x^2 - 3$

21. $f(x) = 8x^2 - 12x - 3$

22. $f(x) = 3x^4 + 2x^3 - x + 15$

USING SYNTHETIC DIVISION **Use synthetic division to decide which of the following are zeros of the function: 1, −1, 2, −2.**

23. $f(x) = x^3 + 7x^2 - 4x - 28$

24. $f(x) = x^3 + 5x^2 + 2x - 8$

25. $f(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$

26. $f(x) = 2x^4 - 9x^3 + 8x^2 + 9x - 10$

27. $f(x) = x^4 + 3x^3 + 3x^2 - 3x - 4$

28. $f(x) = 3x^4 + 3x^3 + 2x^2 + 5x - 10$

29. $f(x) = x^3 - 3x^2 + 4x - 12$

30. $f(x) = x^3 + x^2 - 11x + 10$

31. $f(x) = x^6 - 2x^4 - 11x^2 + 12$

32. $f(x) = x^5 - x^4 - 2x^3 - x^2 + x + 2$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 15–32
Example 2: Exs. 33–58
Example 3: Exs. 59–64

FINDING REAL ZEROS **Find all the real zeros of the function.**

33. $f(x) = x^3 - 8x^2 - 23x + 30$

34. $f(x) = x^3 + 2x^2 - 11x - 12$

35. $f(x) = x^3 - 7x^2 + 2x + 40$

36. $f(x) = x^3 + x^2 - 2x - 2$

37. $f(x) = x^3 + 72 - 5x^2 - 18x$

38. $f(x) = x^3 + 9x^2 - 4x - 36$

39. $f(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$

40. $f(x) = x^4 + x^3 + x^2 - 9x - 10$

41. $f(x) = x^4 + x^3 - 11x^2 - 9x + 18$

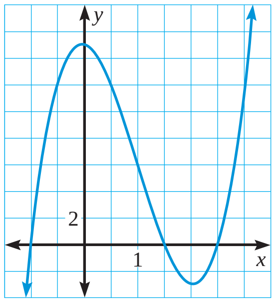
42. $f(x) = x^4 - 3x^3 + 6x^2 - 2x - 12$

43. $f(x) = x^5 + x^4 - 9x^3 - 5x^2 - 36$

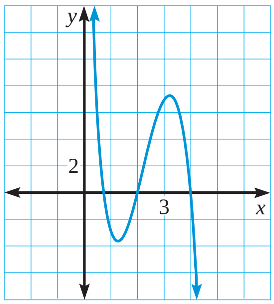
44. $f(x) = x^5 - x^4 - 7x^3 + 11x^2 - 8x + 12$

ELIMINATING POSSIBLE ZEROS **Use the graph to shorten the list of possible rational zeros. Then find all the real zeros of the function.**

45. $f(x) = 4x^3 - 12x^2 - x + 15$



46. $f(x) = -3x^3 + 20x^2 - 36x + 16$



STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with problem solving in Ex. 60.

FINDING REAL ZEROS **Find all the real zeros of the function.**

47. $f(x) = 2x^3 + 4x^2 - 2x - 4$

48. $f(x) = 2x^3 - 5x^2 - 14x + 8$

49. $f(x) = 2x^3 - 5x^2 - x + 6$

50. $f(x) = 2x^3 + x^2 - 50x - 25$

51. $f(x) = 2x^3 - x^2 - 32x + 16$

52. $f(x) = 3x^3 + 12x^2 + 3x - 18$

53. $f(x) = 2x^4 + 3x^3 - 3x^2 + 3x - 5$

54. $f(x) = 3x^4 - 8x^3 - 5x^2 + 16x - 5$

55. $f(x) = 2x^4 + x^3 - x^2 - x - 1$

56. $f(x) = 3x^4 + 11x^3 + 11x^2 + x - 2$

57. $f(x) = 2x^5 + x^4 - 32x - 16$

58. $f(x) = 3x^5 + x^4 - 243x - 81$

FOCUS ON APPLICATIONS

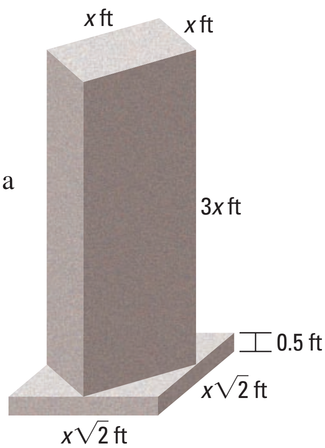
MOLTEN GLASS In order for glass to melt so that it can be poured into a mold, it must be heated to temperatures between 1000°C and 2000°C.

59. **HEALTH PRODUCT SALES** From 1990 to 1994, the mail order sales of health products in the United States can be modeled by

$$S = 10t^3 + 115t^2 + 25t + 2505$$

where S is the sales (in millions of dollars) and t is the number of years since 1990. In what year were about \$3885 million of health products sold? (*Hint:* First substitute 3885 for S, then divide both sides by 5.)

60. **MONUMENTS** You are designing a monument and a base as shown at the right. You will use 90 cubic feet of concrete for both pieces. Find the value of x.



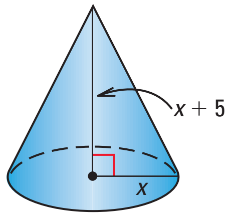
Ex. 60

61. **MOLTEN GLASS** At a factory, molten glass is poured into molds to make paperweights. Each mold is a rectangular prism whose height is 3 inches greater than the length of each side of the square base. A machine pours 20 cubic inches of liquid glass into each mold. What are the dimensions of the mold?

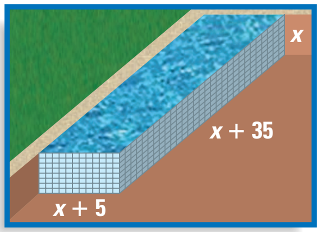
FOCUS ON APPLICATIONS

REAL LIFE **SAND SCULPTURE** The tallest sand sculptures built were over 20 feet tall and each consisted of hundreds of tons of sand.

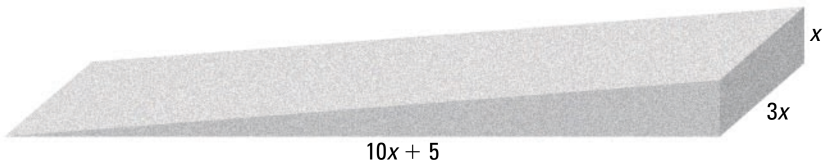
62. **SAND CASTLES** You are designing a kit for making sand castles. You want one of the molds to be a cone that will hold 48π cubic inches of sand. What should the dimensions of the cone be if you want the height to be 5 inches more than the radius of the base?



63. **SWIMMING POOLS** You are designing an in-ground lap swimming pool with a volume of 2000 cubic feet. The width of the pool should be 5 feet more than the depth, and the length should be 35 feet more than the depth. What should the dimensions of the pool be?



64. **WHEELCHAIR RAMPS** You are building a solid concrete wheelchair ramp. The width of the ramp is three times the height, and the length is 5 feet more than 10 times the height. If 150 cubic feet of concrete is used, what are the dimensions of the ramp?



Test Preparation

QUANTITATIVE COMPARISON **In Exercises 65 and 66, choose the statement that is true about the given quantities.**

(A) The quantity in column A is greater.
(B) The quantity in column B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the given information.

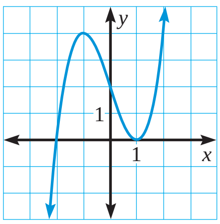
	Column A	Column B
65.	The number of possible rational zeros of $f(x) = x^4 - 3x^2 + 5x + 12$	The number of possible rational zeros of $f(x) = x^2 - 13x + 20$
66.	The greatest real zero of $f(x) = x^3 + 2x^2 - 5x - 6$	The greatest real zero of $f(x) = x^4 + 3x^3 - 2x^2 - 6x + 4$

★ Challenge

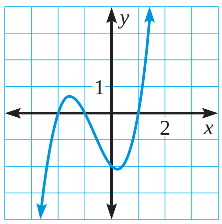
Find the real zeros of the function. Then match each function with its graph.

67. $f(x) = x^3 + 2x^2 - x - 2$ **68.** $g(x) = x^3 - 3x + 2$ **69.** $h(x) = x^3 - x^2 + 2$

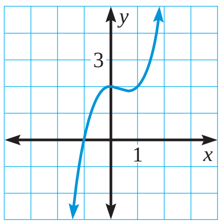
A.



B.



C.



EXTRA CHALLENGE
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70. **CRITICAL THINKING** Is it possible for a cubic function to have more than three real zeros? Is it possible for a cubic function to have no real zeros? Explain.

MIXED REVIEW

SOLVING QUADRATIC EQUATIONS **Solve the equation.** (Review 5.2 for 6.7)

71. $x^2 - 6x + 9 = 0$

72. $x^2 + 12 = 10x - 13$

73. $x - 1 = x^2 - x$

74. $x^2 + 18 = 12x - x^2$

75. $2x^2 - 20x = x^2 - 100$

76. $x^2 - 12x + 49 = 6x - 32$

WRITING QUADRATIC FUNCTIONS **Write a quadratic function in intercept form whose graph has the given *x*-intercepts and passes through the given point.** (Review 5.8 for 6.7)

77. x-intercepts: $-3, 3$
point: $(0, 5)$

78. x-intercepts: $-5, 1$
point: $(-2, -6)$

79. x-intercepts: $-1, 5$
point: $(0, 10)$

80. x-intercepts: $12, 7$
point: $(-11, 7)$

81. x-intercepts: $-12, -6$
point: $(9, -5)$

82. x-intercepts: $2, 8$
point: $(3, -4)$

83. x-intercepts: $4, 10$
point: $(7, 3)$

84. x-intercepts: $-6, 0$
point: $(2, 16)$

85. x-intercepts: $-9, -1$
point: $(1, 20)$

86. **PICTURE FRAMES** You have a picture that you want to frame, but first you have to put a mat around it. The picture is 12 inches by 16 inches. The area of the mat is 204 square inches. If the mat extends beyond the picture the same amount in each direction, what will the final dimensions of the picture and mat be? (Review 5.2)

QUIZ 2

Self-Test for Lessons 6.4–6.6

Factor the polynomial. (Lesson 6.4)

1. $5x^3 + 135$

2. $6x^3 + 12x^2 + 12x + 24$

3. $4x^5 - 16x$

4. $3x^3 - x^2 - 15x + 5$

Find the real-number solutions of the equation. (Lesson 6.4)

5. $7x^4 = 252x^2$

6. $16x^6 = 54x^3$

7. $6x^5 - 18x^4 + 12x^3 = 36x^2$

8. $2x^3 + 5x^2 = 8x + 20$

Divide. Use synthetic division when possible. (Lesson 6.5)

9. $(x^2 + 7x - 44) \div (x - 4)$

10. $(3x^2 - 8x + 20) \div (3x + 2)$

11. $(4x^3 - 7x^2 - x + 10) \div (x^2 - 3)$

12. $(12x^4 + 5x^3 + 3x^2 - 5) \div (x + 1)$

13. $(x^4 + 2x^2 + 3x + 6) \div (x^3 - 3)$

14. $(5x^4 + 2x^3 - x - 5) \div (x + 5)$

Find all the real zeros of the function. (Lesson 6.6)

15. $f(x) = x^3 - 4x^2 - 7x + 28$

16. $f(x) = x^3 - 6x^2 + 21x - 26$

17. $f(x) = 2x^3 + 15x^2 + 22x - 15$

18. $f(x) = 2x^3 + 7x^2 - 28x + 12$

19. **DESIGNING A PATIO** You are a landscape artist designing a patio. The square patio floor is to be made from 128 cubic feet of concrete. The thickness of the floor is 15.5 feet less than each side length of the patio. What are the dimensions of the patio floor? (Lesson 6.6)