Chapter 6A - Exponential and Logarithmic Equations

Exponential Equations

In previous chapters we learned about the exponential and logarithmic functions, studied some of their properties, and learned some of their applications. In this chapter we show how to solve some simple equations which contain the unknown either as an exponent (exponential equation) or as the argument of a logarithmic function.

As a general rule of thumb, to solve an exponential equation proceed as follows:

1. Isolate the expression containing the exponent on one side of the equation.
2. Take the logarithm of both sides to "bring down the exponent".
3. Solve for the variable.

Example 1: Solve $3^x = 25$

Solution:

$$
\begin{aligned}
3^x &= 25 && \text{take the natural log of both sides} \\
x \ln 3 &= \ln 25 && \text{solve for } x \\
x &= \frac{\ln 25}{\ln 3} \\
&\approx 2.929\,947
\end{aligned}
$$

Example 2: Solve $4 + 3^{x+1} = 8$

Solution:

$$
\begin{aligned}
4 + 3^{x+1} &= 8 && \text{isolate } x \\
3^{x+1} &= 4 && \text{take the natural log of both sides} \\
(x+1) \ln 3 &= \ln 4 && \text{solve for } x \\
x &= \frac{\ln 4}{\ln 3} - 1 \\
&\approx .261\,859\,5
\end{aligned}
$$

Example 3: Solve the equation $\dfrac{10}{1 + e^{-x}} = 2$

Solution We need to "isolate" the terms involving x on one side of the equation. We can do this by cross multilpying and then solving for e^{-x}:

$$
\begin{aligned}
1 + e^{-x} &= 5 \\
e^{-x} &= 4 \\
-x &= \ln 4 \\
x &= -\ln 4 \approx -1.386\,294
\end{aligned}
$$

Example 4:   Solve the equation $x^2 2^x - 2^x = 0$.

Solution: This looks slightly difficult. However, let's factor the 2^x term out of the left hand side.

$$x^2 2^x - 2^x = 0$$

$$2^x(x^2 - 1) = 0$$

Since a product can equal zero if and only if one of the factors is zero, we know that if x is a solution, then either $2^x = 0$ or $x^2 - 1 = 0$. But 2^x is never 0, thus, our solution must satisfy

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

Example 5:   Solve the equation $e^{2x} - 3e^x + 2 = 0$.

Solution: This equation really looks hard, and it is until we notice that it is a quadratic equation in e^x. To see that this is the case, set $u = e^x$, then the equation $e^{2x} - 3e^x + 2 = 0$ can be written as $u^2 - 3u + 2$. Solving this latter equation we have

$$(e^x)^2 - 3(e^x) + 2 = 0$$

$$u^2 - 3u + 2 = 0$$

$$(u - 1)(u - 2) = 0$$

Thus, we have $u = 1$ or $u = 2$. In terms of e^x, this means

$$
\begin{array}{lll|lll}
e^x & = & 1 & e^x & = & 2 \\
x & = & \ln 1 & x & = & \ln 2 \\
x & = & 0 & x & \approx & .693\,147\,2
\end{array}
$$
or

Logarithmic Equations

In the previous page we showed how to solve some exponential equations. Here we solve some logarithmic equations.

To solve a logarithmic equation proceed as follows

1. Isolate the expression containing the logarithm on one side of the equation.
2. Exponeniate both sides to remove the log function.
3. Solve for the variable.

Example 1:   Solve $\log x = 35$ for x.

Solution: The main item we need to note here is that log represents the logarithm of a number to base 10. Thus, we need to raise both sides of the equation to the 10th power.

$$\log x = 35$$
$$x = 10^{\log x} = 10^{35}$$

Example 2:   Solve $\ln(x - 3) = 5$ for x.

Solution: For this equation the logarithm used is the natural log. That is, to the base $e \approx 2.718282$.

$$\ln(x - 3) = 5$$
$$x - 3 = e^5$$
$$x = e^5 + 3$$
$$\approx 151.4132$$

Example 3:   Solve $6 - \log_5(3x - 2) = 4$ for x.

Solution:

$$6 - \log_5(3x - 2) = 4$$
$$\log_5(3x - 2) = 6 - 4$$
$$3x - 2 = 5^2$$
$$3x = 25 + 2$$
$$x = \frac{27}{3} = 9$$

Example 4:   Solve the equation $\log_2 3 + \log_2 x = \log_2 5 + \log_2(x - 2)$

Solution: The first thing to do is to use the algebraic properties of log functions to try to simplify this equation.

$$\log_2 3 + \log_2 x = \log_2 5 + \log_2(x - 2)$$
$$\log_2(3x) = \log_2 5(x - 2) \qquad \text{now raise both sides to the power 2.}$$
$$3x = 5(x - 2) = 5x - 10$$
$$2x = 10$$
$$x = 5$$

Example 5:   Solve $\log x + \log(x - 1) = \log(4x)$.

Solution: Here as in Example 4, we first simplify this equation by using some of the logarithm's properties.

$$\log x + \log(x - 1) = \log(4x)$$
$$\log[x(x - 1)] = \log(4x)$$
$$x(x - 1) = 4x$$
$$x^2 - 5x = 0$$
$$x(x - 5) = 0$$

The solutions to this last equation are $x = 0$ and $x = 5$. However, we need to be sure that they are solutions to the original logarithmic equation. There is no problem with the solution $x = 5$, but $x = 0$ is not a valid solution as the term $\log 0$ is not defined.

Hence the only solution to the equation $\log x + \log(x - 1) = \log(4x)$ is $x = 5$.

Exercises for Chapter 6A - Exponential and Logarithmic Equations

For problems 1-16, Solve the equation for x.

1. $3^x = 14$
2. $5e^x = 22$
3. $7(10)^{3x-1} = 5$
4. $2e^{3x-5} = 7$
5. $\dfrac{15}{1 + e^{-2x+1}} = 4$
6. $200(1.02)^{3t} = 1000$
7. $x^2 e^x + 5xe^x - 6e^x = 0$
8. $\ln(4x - 5) = 0$
9. $3 - \log_2(x - 1) = 0$
10. $\log(x^2 - 3x) = 1$
11. $\log_3(2x + 3) = 4$
12. $\log_3 x + \log_3(x + 6) = 3$
13. $1 + \log(3x - 1) = \log(2x + 1)$
14. $\log_2(x^2 - x - 2) = 2$
15. $\ln(\ln x) = 3$
16. $\log(3x - 10) = 2 + \log(x - 2)$

Answers to Exercises for Chapter 6A - Exponential and Logarithmic Equations

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1.

$$3^x = 14$$
$$x = \log_3 14$$
$$\approx 2.402\,174$$

2.

$$5e^x = 22$$
$$x = \ln \frac{22}{5}$$
$$\approx 1.481\,605$$

3.

$$(10)^{3x-1} = \frac{5}{7}$$
$$(3x - 1)\ln 10 = \ln\left(\frac{5}{7}\right)$$
$$3x - 1 = \frac{\ln(\frac{5}{7})}{\ln 10}$$
$$3x = \frac{\ln(\frac{5}{7})}{\ln 10} + 1$$
$$x = \frac{1}{3}\left(\frac{\ln(\frac{5}{7})}{\ln 10} + 1\right)$$
$$\approx 0.284624$$

4.

$$2e^{3x-5} = 7$$
$$x = \frac{5}{3} + \frac{1}{3}\ln\frac{7}{2}$$
$$\approx 2.084\,254$$

5.

$$\frac{15}{1 + e^{-2x+1}} = 4$$
$$4 + 4e^{-2x+1} = 15$$
$$e^{-2x+1} = \frac{11}{4}$$
$$-2x + 1 = \ln\frac{11}{4}$$
$$x = \frac{-1}{2}\left(\ln\frac{11}{4} - 1\right)$$
$$\approx -0.005800$$

6.

$$200(1.02)^{3t} = 1000$$
$$t \approx 27.091\,32$$

7. The given equation $x^2 e^x + 5xe^x - 6e^x = 0$ imples that the following equation is valid.

(Divide by e^x which is never 0.)

$$x^2 + 5x - 6 = 0.$$

The roots of this last equation are $x = -6$ and $x = 1$.

8.

$$\ln(4x - 5) = 0$$
$$4x - 5 = 1$$
$$x = \frac{3}{2}$$

9.

$$3 - \log_2(x - 1) = 0$$
$$\log_2(x - 1) = 3$$
$$x - 1 = 2^3$$
$$x = 9$$

10.

$$\log(x^2 - 3x) = 1$$
$$x^2 - 3x = 10^1$$
$$x^2 - 3x - 10 = 0$$

This last equation has solutions $x = 5$ and $x = -2$. Both of which are solutions to the original equation.

11.

$$\log_3(2x + 3) = 4$$
$$2x + 3 = 3^4$$
$$2x = 78$$
$$x = 39$$

12.

$$\log_3 x + \log_3(x + 6) = 3$$
$$\log_3[x(x + 6)] = 3$$
$$x(x + 6) = 3^3$$
$$x^2 + 6x - 27 = 0$$
$$(x + 9)(x - 3) = 0$$

Solutions to last equation are $x = -9$ and $x = 3$. However, $x = -9$ is not a solution to the original equation since it is not in the domain. Thus, $x = 3$ is the only solution to the original equation.

13.

$$1 + \log(3x - 1) = \log(2x + 1)$$
$$\log\left(\frac{2x + 1}{3x - 1}\right) = 1$$
$$\frac{2x + 1}{3x - 1} = 10$$

The last equation has $x = \frac{11}{28}$ as a solution. $\frac{11}{28}$ is also a solution to the original equation.

14.

$$\log_2(x^2 - x - 2) = 2$$
$$x^2 - x - 2 = 4$$
$$x^2 - x - 6 = 0$$

Solutions to the last equation are $x = 3$ and $x = -2$. Both of them also solve the original equation.

15.

$$\ln(\ln x) = 3$$
$$\ln x = e^3$$
$$x = e^{(e^3)}$$

16.

$$\log(3x - 10) = 2 + \log(x - 2)$$
$$\log \frac{3x - 10}{x - 2} = 2$$
$$\frac{3x - 10}{x - 2} = 100$$

The solution to the last equation is $x = \frac{190}{97}$. However, it is not a solution to the original equation since it is not in the domain.

Chapter 6B - Applications of Exponentials and Logarithms

Exponential Functions and Population Models

There are many species of plants and animals whose populations follow an exponential growth law. We will look at several examples of such behavior in this section.

> A population of some species satisfies an exponential growth law if there are numbers a and k such that if $P(t)$ equals the population of the species at t, then
> $$P(t) = P(0)a^{kt},$$
> where $P(0)$ represents the population at time $t = 0$.

Note: in practice the separate values of a and k are not important. What is crucial is a^k, for if we know this number, then we can compute $P(t)$. Since we can write $a = e^{\ln a}$ every exponential growth law can also be expressed in terms of the natural exponential function. That is,
$$P(t) = P(0)a^{kt} = P(0)e^{kt\ln a}.$$

Example 1:   If $P(t) = 6 \cdot 5^{2t}$, then $P(t)$ satisfies an exponential growth law. What is $P(0)$. Find a value of t such that $P(t) = 150$.

Solution: To find out what $P(0)$ equals we set $t = 0$ in the expression for $P(t)$.
$$P(0) = 6 \cdot 5^0 = 6 \cdot 1 = 6.$$

The last part of the example is to find a value of t for which $P(t) = 150$.
$$150 = P(t) = 6 \cdot 5^{2t} \implies$$
$$\frac{150}{6} = 25 = 5^{2t} \implies$$
$$25 = (5^2)^t = 25^t \text{ a solution to this equation is}$$
$$t = 1$$

Example 2:   Suppose that a bacterial colony on a petri dish doubles its population every 3 hours. Show that the number of bacteria satisfies an exponential growth law.

Solution: Let $P(t)$ represent the number of bacteria present at time t in hours. The statement that the number of bacteria doubles every 3 hours can be written as $P(t + 3) = 2P(t)$. The formulas below are constructed using this equation.
$$P(3) = 2P(0)$$
$$P(6) = P(3 + 3) = 2P(3) = 2[2P(0)] = 2^2 P(0)$$
$$P(9) = P(6 + 3) = 2P(6) = 2[2^2 P(0)] = 2^3 P(0) \quad \text{Do you see a relationship}$$
$$P(12) = P(9 + 3) = 2P(9) = 2[2^3 P(0)] = 2^4 P(0) \quad \text{between the argument of } P$$
$$P(15) = P(12 + 3) = 2P(12) = 2[2^4 P(0)] = 2^5 P(0) \quad \text{and the exponent of } 2?$$

There is a relationship between the argument of $P(t)$ and the exponent of 2. If t is the argument of P, then the exponent of 2 is $t/3$. We conjecture the following formula.
$$P(t) = P(0)2^{t/3}.$$

Let's verify that this function satisfies the condition that every three hours it's size doubles:

$$P(t+3) = P(0)2^{(t+3)/3} = P(0)2^{t/3+1} = P(0)2^{t/3}2 = 2[P(0)2^{t/3}] = 2P(t)$$

Thus, we have found constants a and k such that $P(t) = P(0)a^{kt}$, where $a = 2$ and $k = 1/3$. Hence the bacterial population satisfies an exponential growth law.

Question: Which of the following functions satisfy an exponential growth law? (Hint: more than one of these functions satisfies an exponential growth law.)

a) $2t^3$

b) $\dfrac{2t^2 - 5}{t + 1}$

c) 2^{-t}

d) $\dfrac{3}{5^t}$

e) $15(56)^{5t}$

Answer:

a) This function does not satisfy an exponential growth law.
b) Not an exponential growth law.
c) This is an exponential growth law. $P(0) = 1$, $a = 2$, and $k = -1$.
d) This is an exponential growth law. $P(0) = 3$, $a = 5$, and $k = -1$
e) This is an exponential growth law. $P(0) = 15$, $a = 56$, and $k = 5$.

Question: Express $5 * 4^{kt}$ in terms of the natural exponential function.
Answer:
$$5 * 4^{kt} = 5 * (e^{\ln 4})^{kt}$$
$$\approx 5 * (e^{1.39})^{kt}$$
$$= 5 * e^{1.39kt}$$

Example 3: Let $P(t) = 35 \cdot 2^{3t}$. What do $P(0)$, $P(1)$, and $P(3)$ equal?

Solution: To answer these questions we only need to evaluate the function $P(t)$ at the specified values of t.
$$P(0) = 35 \cdot 2^0 = 35$$
$$P(1) = 35 \cdot 2^1 = 70$$
$$P(3) = 35 \cdot 2^{3 \cdot 3} = 35 \cdot 2^9 = 35 \cdot 512 = 17,920$$

Example 4:   Suppose $P(t)$ satisfies an exponential growth law. If $\dfrac{P(2)}{P(1)} = 5$, what must a^k equal? If $P(0) = 6$, determine $P(4)$.

Solution: Since $P(t) = P(0)a^{kt}$, we know that $\dfrac{P(2)}{P(1)} = \dfrac{P(0)a^{2k}}{P(0)a^k} = a^k$. Since we are told this ratio equals 5, we have $a^k = 5$. To calculate $P(4)$ we have

$$P(4) = P(0)a^{4k}$$
$$= 6(a^k)^4$$
$$= 6(5)^4$$
$$= 3750$$

Example 5:   A biologist counts the number of bacteria in a petri dish every 3 hours. The table below gives the data she found. Assuming the population of the bacteria satisfies an exponential growth law, use the data to determine the precise law. That is find a, k, and $P(0)$. Hint: it is only necessary to determine a^k. The values of a and k by themselves are not needed to compute $P(t)$.

t	0	3	6	9	12
$P(t)$	6.7	8.92	11.87	15.79	21.03

The population $P(t)$ is in hundreds. Thus, 6.7 represents 670 bacteria.

Solution: $P(0)$ can be read right from the table. $P(0) = 6.7$. Since we are assuming that the population of the bacteria satisfies an exponential growth law we are assuming that $P(t) = 6.7a^{kt} = 6.7(a^k)^t$. If we look at the ratios of the tabulated data we have the following.

$$\frac{8.92}{6.7} = \frac{P(3)}{P(0)} = \frac{6.7(a^k)^3}{6.7} = (a^k)^3$$

Thus, we should have $(a^k)^3 = \dfrac{8.92}{6.7} \approx 1.3313433$, or $a^k \approx (1.3313433)^{1/3} \approx 1.1000946$. Let's look at some of the other ratios.

$$(a^k)^3 = \frac{P(6)}{P(3)} = \frac{11.87}{8.92} \approx 1.3307175 \implies$$
$$a^k \approx (1.3307175)^{1/3} \approx 1.0999222$$

This is pretty good agreement with the first estimate of a^k. For one last comparison let's look at the ratio of $\dfrac{P(12)}{P(3)}$.

$$\frac{21.03}{8.92} = \frac{P(12)}{P(3)} = \frac{(a^k)^{12}}{(a^k)^3} = (a^k)^9$$

Thus, we should have $(a^k) \approx \left(\frac{21.03}{8.92} \right)^{1/9} \approx 1.0999832$. Still in very good agreement with our first two calculations. Thus, to one decimal place we estimate that $a^k = 1.1$.

Question: If we use the ratios $\frac{P(9)}{P(3)}$, what would we get for an estimate of a^k?

Answer: $\frac{P(9)}{P(3)} = \frac{P(0)(a^k)^9}{P(0)(a^k)^3} = (a^k)^6$. From the table we have $\frac{P(9)}{P(3)} = \frac{15.79}{8.92} \approx 1.7701794$.

Thus,

$$a^k \approx (1.7701794)^{1/6}$$
$$\approx 1.099857$$

Example 6: A biologist decides that an epidemic spreads through a population of a city according to the following model $p(t) = 1 - e^{-0.34t}$, where $p(t)$ represents that fraction of the city's population which has come down with the disease, and t is in weeks. How long will it take for 90% of the city to become infected?

Solution: Notice that $p(0) = 0$. That is, at the beginning of the epidemic no one in the city has the disease. Note too, that as time progressess a larger and larger fraction of the city becomes infected. In fact the value of $p(t)$ gets closer and closer to 1 as t gets larger and larger. The equation we need to solve is

$$.9 = 1 - e^{-0.3t}$$
$$e^{-0.3t} = 1 - 0.9 = 0.1$$
$$-0.3t = \ln(0.1)$$
$$t = \frac{\ln(0.1)}{-0.3}$$
$$\approx 7.67528$$

It seems that this is a disease which spreads very rapidly. After 8 weeks over 90% of the population is infected.

Exponential Functions and Radioactive Decay

There are many material substances which decay radioactively. That is, they spontaneously change into a different material, and in the decay process emit charged particles. Some naturally occurring isotopes which decay are carbon 14, ^{14}C, uranium 234, ^{234}U, and mercury 196, ^{196}Hg. Associated with any radioactive substance is a period of time called its **half-life**. The **half-life** of a substance is how long it takes for half of the substance to decay.

Thus, if the half life of a substance is 2 years, and we start out with one pound of the material, then after 2 years we'll have 1/2 pound left, and after 4 years we'll have 1/2 of 1/2 or 1/4 of a pound left, etc. The table below lists some radioactive elements, their chemical symbol, and their half-life.

Element	carbon 14	platinum 192	radium 226	tungston 183	uranium 235
Symbol	^{14}C	^{192}Pt	^{226}Ra	^{183}W	^{235}U
Half-life(years)	5.8×10^3	10^5	$1,622$	10^{17}	7.1×10^8

If an element decays radioactively, then the amount of this element at any time t satisfies an exponential growth/decay law. That is, if $A(t)$ denotes the amount of material at time t, then

$$A(t) = A(0)e^{kt}.$$

The difference between exponential functions used to model interest earned, population growth, and radioactive decay is that, in the first two, the term (e^k) is larger than 1 while in a decay situation the term (e^k) is less than 1.

Example 1:   Using the fact that the half-life of carbon 14 is 5800 years, determine the exponential growth/decay law which ^{14}C satisfies.

Solution: Let $A(0)$ denote the amount of ^{14}C present at $t = 0$. Let t_2 denote the half-life. Then we have $A(t_2) = \frac{1}{2}A(0)$. Using the formula $A(t) = A(0)e^{kt}$, we have

$$\frac{1}{2}A(0) = A(t_2) = A(0)e^{k*t_2} \quad \text{divide by } A(0)$$

$$\frac{1}{2} = e^{k*t_2} \quad \text{take the natural log of both sides}$$

$$- \ln 2 = t_2 * k$$

$$k = \frac{-\ln 2}{t_2}$$

So, for an element with a half life of t_2 years, its exponential growth law is $A(t) = A(0)e^{-t\ln 2/t_2}$. Thus, since $t_2 = 5800$ for ^{14}C, this radioactive element satisfies the law $A(t) = A(0)e^{-t\ln 2/5800}$.

Question: If the half life of a substance is 5 years, how many years will it take for 2 pounds of this substance to decay to $\frac{1}{8}$ of a pound ? Hint: you do not need to determine the exponential decay law.

Answer: 20 years is correct. After 5 years, 1 pound is left. After 10 years, 1/2 pound is left. After 15 years, 1/4 pound is left. After 20 years, 1/8 pound left.

Example 2:   The half life of uranium 235 is 7.1×10^8 years. If we start out with 1.5 kilograms of ^{235}U in 1999, how much uranium will be left after $10,000$ years?

Solution: We saw on the preceding page that the exponential growth/decay law is

$$A(t) = A(0)e^{-t\ln 2/t_2},$$

where t_2 is the half-life. Thus, for 1.5 kilogram of ^{235}U we have

$$A(t) = 1.5e^{-t\ln 2/(7.1*10^8)}.$$

So after $10,000$ years we will have

$$A(10,000) = 1.5e^{-10000\ln 2/(7.1*10^8)}$$

$$\approx 1.499 \text{ kilograms.}$$

Not much ^{235}U has decayed after $10,000$ years.

Example 3:   Suppose a radioactive substance satisfies the exponential growth/decay law $A(t) = A(0)4^{-t}$, where t is in centuries. What is the half-life of this substance?

Solution: We want to find that value of t for which $A(t) = \frac{1}{2}A(0)$. That is,

$$\frac{1}{2}A(0) = A(0)4^{-t} \Rightarrow$$

$$\frac{1}{2} = 4^{-t}$$

To solve this equation we take the natural log of both sides.

$$\frac{1}{2} = 4^{-t}$$

$$-\ln 2 = -t\ln 4$$

$$t = \frac{-\ln 2}{-\ln 4} = \frac{\ln 2}{2\ln 2} = \frac{1}{2}$$

Thus, the half-life of this substance equals $\frac{1}{2}$ century or 50 years.

Example 4:   A physicist compiles the following table of data for the decay of a radioactive material. Assuming the material satisfies an exponential decay law, find an exponential function which models the data.

time in months	amount of material in ounces
4	15.3726
8	14.7699
12	14.1907

Solution: The function we use to model this data has the form $f(t) = ca^{kt}$, where c, a, and k are constants to be determined. However, we can essentially ignore what the base is, because we now realize that we can use the natural exponential function to model any form of exponential growth. That is, we look for a function of the form $f(t) = ce^{kt}$, where c and k have to be determined. The first two rows in the above table lead to the following equations

$$15.3726 = ce^{4k}$$

$$14.7699 = ce^{8k}.$$

Taking the natural log of both sides of each equation we have

$$\ln 15.3726 = \ln c + 4k$$

$$\ln 14.7699 = \ln c + 8k$$

Subtracting the second equation from the first we get

$$\ln 15.5726 - \ln 14.7699 = \ln c + 4k - (\ln c + 8k)$$

$$\ln\left(\frac{15.3726}{14.7699}\right) = -4k$$

$$k = \frac{\ln\left(\frac{15.3726}{14.7699}\right)}{-4}$$

$$k \approx -9.99884 \times 10^{-3}$$

We now take this value for k and substitute into the first equation and then solve for c.

$$\ln 15.3726 = \ln c + 4(-9.99884 \times 10^{-3})$$

$$\ln c = \ln 15.3726 - 4(-9.99884 \times 10^{-3})$$

$$\ln c \approx 2.77258 \quad \text{now exponentiate both sides}$$

$$c \approx e^{2.77258} \approx 15.9999$$

Thus, the exponential function which models the given data equals

$$f(t) \approx 16e^{-0.001t},$$

where we have rounded off the values of c and k.

Remember that t has units of months and $f(t)$ has units of ounces.

Example 5:   What is the half-life of this material.

Solution: We are looking for a value of t for which $f(t) = \frac{1}{2}f(0)$. This leads to the equation.

$$16e^{-0.001t} = f(t) = \frac{1}{2}f(0) = \frac{1}{2}16 = 8$$

$$e^{-0.001t} = \frac{8}{16} = \frac{1}{2} \quad \text{take the natural log of both sides}$$

$$-0.001t = \ln 0.5$$

$$t = \frac{\ln 0.5}{-0.001} = 693.147$$

Thus, the half-life of this material is approximately 693 months or a little less than 58 years.

Exercises for Chapter 6B - Applications of Exponentials and Logarithms

1. A certain strain of bacteria satisfies the exponential growth law $P(t) = 15 \cdot 4^t$, where t is in hours. Calculate the number of bacteria at 1 hour intervals for the first 6 hours.

2. A chemist and a biologist want to test if a certain chemical is effective in controlling a particular bacteria. A specific colony of this bacteria satisfies the exponential growth law $P(t) = 100(4.5)^t$, where t is in hours. At time $t = 0$ the two scientists expose the colony to the chemical which they hope will control the bacteria. The biologist, at hourly intervals, counts the number of bacteria. Her data is tabulated below. Do you think the chemical was effective in controlling the bacteria?

t	0	1	2	3	4	5	6
count	102	300	990	2150	4080	8450	16,375

3. An anthropologist while studying a European region is able to determine the population of this region at various times. In so doing he believes that this the population of this region satisfies the exponential growth law $P(t) = 500e^{0.002t}$, where $t = 0$ corresponds to 2500 BC. What does this model predict the population of the region will be in the year 2000?

4. Let $p(t) = 200e^{kt}$ represent the number of bacteria in a petri dish after t days. Suppose the number of bacteria doubles every 5 days. What must k equal?

5. A epidemiologist while studying the progession of a flu epidemic decides that the function $p(t) = \frac{3}{4}(1 - e^{-kt})$, $k > 0$, will be a good model for the fraction of the earth's population which will contract the flu. t is in months. If after 2 months $\frac{1}{1000}$ of the earth's population has the flu, what is the what is the value of k?

6. The half-life of tungston is 10^{17} years. How long will it take for 10 grams of tungston to decay to 5 grams, and 2.5 grams?

7. The half-life of tungston is 10^{17} years. If there is currently a total of 10^{10} pounds of tungston, how much tungston will be left 50,000 years from now?

8. After 5 years 10 pounds of a radioactive material has decayed to 2.5 pounds. What is the half-life of this radioactive material?

9. Refering to the material in the previous problem, how much of the original 10 pounds will be left after 50 years?

10. If a radiactive material satisfies the decay law $A(t) = 15\left(\frac{1}{2}\right)^{4t}$, what is the half-life of this material, and how much will be left in 200 years?

11. Radium 226 has a half-life of $1,622$ years. Radium is mainly used for medical treatments. Suppose a medical center buys 1/4 pound of ^{226}Ra for $5000. What is the dollar value of the radium after 100 years?

12. If $f(x) = 16e^{-3x}$, find x such that $f(x) = 8$. If $f(x)$ represented a radioactive material, then the value of x we are seeking would be called the half-life of the material.

13. Find the value of k such that if $f(t) = ce^{kt}$ represents the amount of radioactive material of a substance after t years, then this substance has a half-life of 1500 years.

14. The number of bacteria present in a culture $N(t)$ at time t hours is given by $3000(2)^t$.
a) What is the initial population?   b) How many bacteria are present in 24 hours?
c) How long will it take the population to triple in size?

15. The mass $m(t)$ remaining after t days from a $40 - g$ sample of thorium-234 is given by $m(t) = 40e^{-0.0277t}$.
a) How much of the sample remains after 60 days?
b) After how long will only 10 g of the sample remain?

Answers to Exercises for Chapter 6B - Applications of Exponentials and Logarithms

1. $P(1) = 60$ $P(2) = 240$ $P(3) = 960$ $P(4) = 3840$ $P(5) = 15360$ $P(6) = 61440$. Thus, after 1 hour there are 60 bacteria. After 2 hours there are 240 bacteria. After 3 hours there are 960 bacteria. After 4 hours there are 3840 bacteria. After 5 hours there are 15360 bacteria. After 6 hours there are 61440 bacteria.

2. Before answering the question as to the efficacy of the chemical, we should see what the exponential model predicts. $P(0) = 100$, $P(1) = 450.0$, $P(2) = 2025.0$, $P(3) = 9112.5$, $P(4) = 41006.25$, $P(5) = 184528.13$. After comparing these numbers to the actual numbers, we certainly feel that the chemcal has inhibited the growth of the bacteria. However, it does appear that the bacterial colony is still experiencing exponential growth, although at a reduced rate. It appears to double in size every hour instead of every half-hour. So the conclusion the scientists should draw is that the chemical slows down the growth of the colony, but the bacteria still grow exponentially.

3. The year 2000 corresponds to $t = 4500$. The model predicts that the population of the region will be

$$P(4500) \approx 4,051,542.$$

4. We are to find k such that $p(5) = 2p(0)$. The equation which this lead to is
$$p(5) = 200e^{5k} = 2p(0)$$
$$= 2 \cdot 200$$
$$e^{5k} = 2$$
$$5k = \ln 2$$
$$k = \frac{\ln 2}{5}$$
$$\approx 0.138629$$

5. The data $p(2) = \frac{1}{1000}$ means that
$$\frac{3}{4}(1 - e^{-2k}) = \frac{1}{1000}$$
$$1 - e^{-2k} = \frac{4}{3000}$$
$$e^{-2k} = \frac{2996}{3000}$$
$$-2k = \ln\left(\frac{2996}{3000}\right)$$
$$k = \frac{-1}{2}\ln\left(\frac{2996}{3000}\right)$$
$$\approx 6.671 \times 10^{-4}$$

6. It will take 10^{17} years for the 10 grams to decay to 5 grams, and it will take another 10^{17} years or a total of $2 \cdot 10^{17}$ years for it to decay to 2.5 grams.

7. If $A(t)$ represents the amount of tungston t years from now, then we know that

$$A(t) = 10^{10}\left(\frac{1}{2}\right)^{t/10^{17}}$$

Thus, $50,000$ years from now there will be

$$A50,000 = 10^{10}\left(\frac{1}{2}\right)^{50000/10^{17}}$$
$$\approx 10^{10}(0.999\,999\,999\,999\,653)$$
$$\approx 9,999,999,999.996\,53 \text{ pounds.}$$

8. The amount of material has decayed to $1/4$ of the original amount. So this means 2 half-lives have passed. Thus, $2t_h = 5$ or

$$t_h = \frac{5}{2} = 2.5 \text{ years.}$$

9. $A(t) = 10\left(\frac{1}{2}\right)^{t/t_h}$. Thus,

$$A(50) = 10\left(\frac{1}{2}\right)^{50/2.5}$$
$$\approx 10\left(\frac{1}{2}\right)^{20.0}$$
$$\approx 9.54 \times 10^{-6} \text{ pounds.}$$

10. From $A(t) = 15\left(\frac{1}{2}\right)^{4t}$ have that

$$\frac{t}{t_h} = 4t.$$

Solving for t_h, we get

$$t_h = \frac{1}{4}.$$

After 200 years

$$A(200) = 15\left(\frac{1}{2}\right)^{800}$$
$$\approx 2.3 \times 10^{-240}$$

pounds will be left.

11. The value of the radium is $20,000$ per pound, since $1/4$ pound cost $5,000$. To determine the dollar value of the radium after 100 years we need to first compute how much radium will be left after 100 years.

$$A(100) = (1/4)\left(\frac{1}{2}\right)^{100/1622}$$
$$\approx .239\,542 \text{ pounds.}$$

Thus, the dollar value of the radium will be

$$20000(.239\,542) \approx \$4790.84$$

12. From $f(x) = 16e^{-3x}$, we get the equation

$$8 = 16e^{-3x} \text{ divide by } 16$$
$$e^{-3x} = \frac{1}{2} \quad \text{take logs}$$
$$-3x = \ln(-1/2)$$
$$x = \frac{\ln 2}{3}$$
$$\approx 0.231\,049$$

13. We want to find a value of k such that $f(1500) = \frac{1}{2}f(0)$. This leads to the equations

$$ce^{1500k} = \frac{1}{2}c$$
$$e^{1500k} = \frac{1}{2}$$
$$1500k = \ln(1/2) = -\ln 2$$
$$k = \frac{-\ln 2}{1500}$$
$$\approx -4.620\,98 \times 10^{-4}$$

14.

    a.   3000

    b. $N(24) = 3000(2)^{24} \approx 5.033\,165 \times 10^{10}$

    c.   The equation we need to solve is

$$3 * 3000 = N(t) = 3000(2)^t$$

    or

$$2^t = 3$$
$$t\ln 2 = \ln 3$$
$$t = \frac{\ln 3}{\ln 2}$$
$$\approx 1.584\,963 \text{ hours}$$

15.

    a. $m(60) \approx 7.590$ grams

    b.   We need to solve the equation

$$10 = m(t)$$

    This leads to the equation

$$10 = 40e^{-0.0277t}$$
$$e^{0.0277t} = 4$$
$$0.0277t = \ln 4$$
$$t = \frac{\ln 4}{0.0277}$$
$$\approx 50.046\,72 \text{ days}$$