2-6 Algebraic Proof

What You'll Learn

- Use algebra to write two-column proofs.
- Use properties of equality in geometry proofs.

How is mathematical evidence similar to evidence in law?

Lawyers develop their cases using logical arguments based on evidence to lead a jury to a conclusion favorable to their case. At the end of a trial, a lawyer will make closing remarks summarizing the evidence and testimony that they feel proves their case. These closing arguments are similar to a proof in mathematics.

Vocabulary

- deductive argument
- two-column proof
- formal proof

ALGEBRAIC PROOF Algebra is a system with sets of numbers, operations, and properties that allow you to perform algebraic operations.

Study Tip

Commutative and Associative Properties
Throughout this text, we shall assume the Commutative and Associative Properties for addition and multiplication.

Concept Summary — Properties of Equality for Real Numbers

Reflexive Property	For every number a, $a = a$.
Symmetric Property	For all numbers a and b, if $a = b$, then $b = a$.
Transitive Property	For all numbers a, b, and c, if $a = b$ and $b = c$, then $a = c$.
Addition and Subtraction Properties	For all numbers a, b, and c, if $a = b$, then $a + c = b + c$ and $a - c = b - c$.
Multiplication and Division Properties	For all numbers a, b, and c, if $a = b$, then $a \cdot c = b \cdot c$ and if $c \neq 0$, $\frac{a}{c} = \frac{b}{c}$.
Substitution Property	For all numbers a and b, if $a = b$, then a may be replaced by b in any equation or expression
Distributive Property	For all numbers a, b, and c, $a(b + c) = ab + ac$.

The properties of equality can be used to justify each step when solving an equation. A group of algebraic steps used to solve problems form a **deductive argument**.

Example 1 Verify Algebraic Relationships

Solve $3(x - 2) = 42$.

Algebraic Steps	**Properties**
$3(x - 2) = 42$	Original equation
$3x - 6 = 42$	Distributive Property
$3x - 6 + 6 = 42 + 6$	Addition Property
$3x = 48$	Substitution Property
$\frac{3x}{3} = \frac{48}{3}$	Division Property
$x = 16$	Substitution Property

Bob Daemmrich/The Image Works

Example 1 is a proof of the conditional statement *If* $5x + 3(x - 2) = 42$, *then* $x = 6$. Notice that the column on the left is a step-by-step process that leads to a solution. The column on the right contains the reason for each statement.

In geometry, a similar format is used to prove conjectures and theorems. A **two-column proof**, or **formal proof**, contains statements and reasons organized in two columns. In a two-column proof, each step is called a *statement*, and the properties that justify each step are called *reasons*.

Example 2 Write a Two-Column Proof

Write a two-column proof.

a. If $3\left(x - \frac{5}{3}\right) = 1$**, then** $x = 2$

Statements	Reasons
1. $3\left(x - \frac{5}{3}\right) = 1$	1. Given
2. $3x - 3\left(\frac{5}{3}\right) = 1$	2. Distributive Property
3. $3x - 5 = 1$	3. Substitution
4. $3x - 5 + 5 = 1 + 5$	4. Addition Property
5. $3x = 6$	5. Substitution
6. $\frac{3x}{3} = \frac{6}{3}$	6. Division Property
7. $x = 2$	7. Substitution

b. Given: $\frac{7}{2} - n = 4 - \frac{1}{2}n$

Prove: $n = -1$

Proof:

Statements	Reasons
1. $\frac{7}{2} - n = 4 - \frac{1}{2}n$	1. Given
2. $2\left(\frac{7}{2} - n\right) = 2\left(4 - \frac{1}{2}n\right)$	2. Multiplication Property
3. $7 - 2n = 8 - n$	3. Distributive Property
4. $7 - 2n + n = 8 - n + n$	4. Addition Property
5. $7 - n = 8$	5. Substitution
6. $7 - n - 7 = 8 - 7$	6. Subtraction Property
7. $-n = 1$	7. Substitution
8. $\frac{-n}{-1} = \frac{1}{-1}$	8. Division Property
9. $n = -1$	9. Substitution

Study Tip

Mental Math

If your teacher permits you to do so, some steps may be eliminated by performing mental calculations. For example, in part **a** of Example 2, statements 4 and 6 could be omitted. Then the reason for statements 5 would be Addition Property and Division Property for statement 7.

GEOMETRIC PROOF Since geometry also uses variables, numbers, and operations, many of the properties of equality used in algebra are also true in geometry. For example, segment measures and angle measures are real numbers, so properties from algebra can be used to discuss their relationships. Some examples of these applications are shown below.

Property	Segments	Angles
Reflexive	$AB = AB$	$m\angle 1 = m\angle 1$
Symmetric	If $AB = CD$, then $CD = AB$.	If $m\angle 1 = m\angle 2$, then $m\angle 2 = m\angle 1$.
Transitive	If $AB = CD$ and $CD = EF$, then $AB = EF$.	If $m\angle 1 = m\angle 2$ and $m\angle 2 = m\angle 3$, then $m\angle 1 = m\angle 3$.

Standardized Test Practice

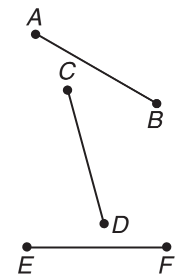
Example 3 Justify Geometric Relationships

Multiple-Choice Test Item

If $\overline{AB} \cong \overline{CD}$, and $\overline{CD} \cong \overline{EF}$, then which of the following is a valid conclusion?

I $AB = CD$ and $CD = EF$
II $\overline{AB} \cong \overline{EF}$
III $AB = EF$

(A) I only (B) I and II
(C) I and III (D) I, II, and III



Test-Taking Tip

More than one statement may be correct. Work through each problem completely before indicating your answer.

Read the Test Item

Determine whether the statements are true based on the given information.

Solve the Test Item

Statement I:

Examine the given information, $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$. From the definition of congruent segments, if $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $AB = CD$ and $CD = EF$. Thus, Statement I is true.

Statement II:

By the definition of congruent segments, if $AB = EF$, then $\overline{AB} \cong \overline{EF}$. Statement II is true also.

Statement III:

If $AB = CD$ and $CD = EF$, then $AB = EF$ by the Transitive Property. Thus, Statement III is true.

Because Statements I, II, and III are true, choice D is correct.

In Example 3, each conclusion was justified using a definition or property. This process is used in geometry to verify and prove statements.

Example 4 Geometric Proof

TIME **On a clock, the angle formed by the hands at 2:00 is a 60° angle. If the angle formed at 2:00 is congruent to the angle formed at 10:00, prove that the angle at 10:00 is a 60° angle.**

Given: $m\angle 2 = 60$
$\angle 2 \cong \angle 10$

Prove: $m\angle 10 = 60$

Proof:

Statements	Reasons
1. $m\angle 2 = 60$ $\angle 2 \cong \angle 10$	**1.** Given
2. $m\angle 2 = m\angle 10$	**2.** Definition of congruent angles
3. $60 = m\angle 10$	**3.** Substitution
4. $m\angle 10 = 60$	**4.** Symmetric Property

Check for Understanding

Concept Check

1. **OPEN ENDED** Write a statement that illustrates the Substitution Property of Equality.
2. **Describe** the parts of a two-column proof.
3. **State** the part of a conditional that is related to the *Given* statement of a proof. What part is related to the *Prove* statement?

Guided Practice

State the property that justifies each statement.

4. If $2x = 5$, then $x = \frac{5}{2}$
5. If $\frac{x}{2} = 7$, then $x = 14$.
6. If $x = 5$ and $b = 5$, then $x = b$.
7. If $XY - AB = WZ - AB$, then $XY = WZ$.
8. Solve $\frac{x}{2} + 4x - 7 = 11$. List the property that justifies each step.
9. Complete the following proof.

Given: $5 - \frac{2}{3}x = 1$

Prove: $x = 6$

Proof:

Statements	Reasons
a. ?	**a.** Given
b. $3\left(5 - \frac{2}{3}x\right) = 3(1)$	**b.** ?
c. $15 - 2x = 3$	**c.** ?
d. ?	**d.** Subtraction Prop.
e. $x = 6$	**e.** ?

PROOF **Write a two-column proof.**

10. Prove that if $25 = -7(y - 3) + 5y$, then $-2 = y$.
11. If rectangle $ABCD$ has side lengths $AD = 3$ and $AB = 10$, then $AC = BD$.
12. The Pythagorean Theorem states that in a right triangle ABC, $c^2 = a^2 + b^2$. Prove that $a = \sqrt{c^2 - b^2}$.

Standardized Test Practice

13. **ALGEBRA** If $8 + x = 12$, then $4 - x =$? .

Ⓐ 28 Ⓑ 24 Ⓒ 0 Ⓓ 4

Practice and Apply

Homework Help

For Exercises	See Examples
15, 16, 20	1
14, 17–19, 21	2
22–27	3
28, 29	4

Extra Practice
See page 757.

State the property that justifies each statement.

14. If $m\angle A = m\angle B$ and $m\angle B = m\angle C$, $m\angle A = m\angle C$.
15. If $HJ + 5 = 20$, then $HJ = 15$.
16. If $XY + 20 = YW$ and $XY + 20 = DT$, then $YW = DT$.
17. If $m\angle 1 + m\angle 2 = 90$ and $m\angle 2 = m\angle 3$, then $m\angle 1 + m\angle 3 = 90$.
18. If $\frac{1}{2}AB = \frac{1}{2}EF$, then $AB = EF$.
19. $AB = AB$

20. If $2\left(x - \frac{3}{2}\right) = 5$, which property can be used to support the statement $2x - 3 = 5$?

21. Which property allows you to state $m\angle 4 = m\angle 5$, if $m\angle 4 = 35$ and $m\angle 5 = 35$?

22. If $\frac{1}{2}AB = \frac{1}{2}CD$, which property can be used to justify the statement $AB = CD$?

23. Which property could be used to support the statement $EF = JK$, given that $EF = GH$ and $GH = JK$?

Complete each proof.

24. **Given:** $\frac{3x + 5}{2} = 7$

Prove: $x = 3$

Proof:

Statements	Reasons
a. $\frac{3x + 5}{2} = 7$	a. ___?___
b. ___?___	b. Mult. Prop.
c. $3x + 5 = 14$	c. ___?___
d. $3x = 9$	d. ___?___
e. ___?___	e. Div. Prop.

25. **Given:** $2x - 7 = \frac{1}{3}x - 2$

Prove: $x = 3$

Proof:

Statements	Reasons
a. ___?___	a. Given
b. ___?___	b. Mult. Prop.
c. $6x - 21 = x - 6$	c. ___?___
d. ___?___	d. Subt. Prop.
e. $5x = 15$	e. ___?___
f. ___?___	f. Div. Prop.

More About. . .

Physics

A gymnast exhibits kinetic energy when performing on the balance beam. The movements and flips show the energy that is being displayed while the gymnast is moving.

Source: www.infoplease.com

PROOF **Write a two-column proof.**

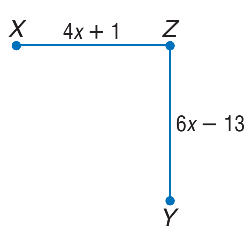
26. If $4 - \frac{1}{2}a = \frac{7}{2} - a$, then $a = -1$.

27. If $-2y + \frac{3}{2} = 8$, then $y = -\frac{13}{4}$.

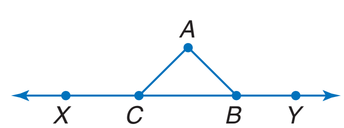
28. If $-\frac{1}{2}m = 9$, then $m = -18$.

29. If $5 - \frac{2}{3}z = 1$, then $z = 6$.

30. If $XZ = ZY$, $XZ = 4x + 1$, and $ZY = 6x - 13$, then $x = 7$.

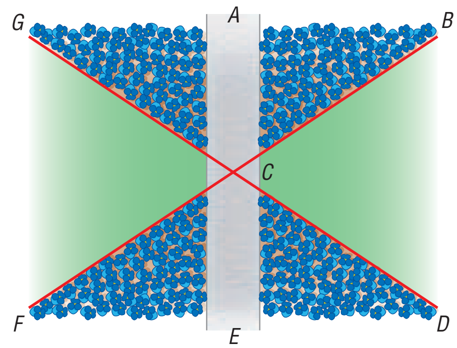


31. If $m\angle ACB = m\angle ABC$, then $m\angle XCA = m\angle YBA$.

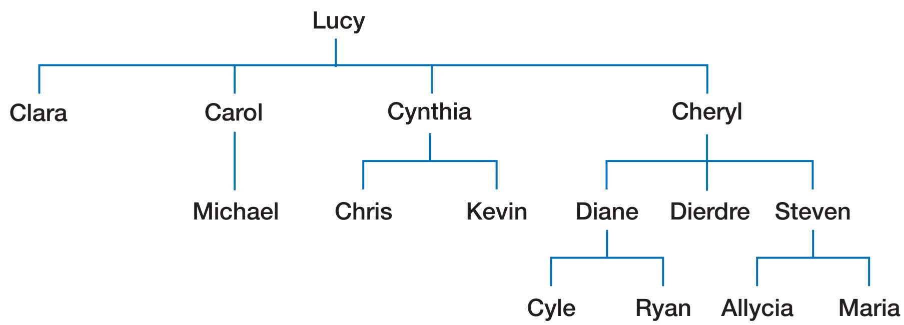


32. **PHYSICS** Kinetic energy is the energy of motion. The formula for kinetic energy is $E_k = h \cdot f + W$, where h represents Planck's Constant, f represents the frequency of its photon, and W represents the work function of the material being used. Solve this formula for f and justify each step.

33. **GARDENING** Areas in the southwest and southeast have cool but mild winters. In these areas, many people plant pansies in October so that they have flowers outside year-round. In the arrangement of pansies shown, the walkway divides the two sections of pansies into four beds that are the same size. If $m\angle ACB = m\angle DCE$, what could you conclude about the relationship among $\angle ACB$, $\angle DCE$, $\angle ECF$, and $\angle ACG$?



CRITICAL THINKING **For Exercises 34 and 35, use the following information.**
Below is a family tree of the Gibbs family. Clara, Carol, Cynthia, and Cheryl are all daughters of Lucy. Because they are sisters, they have a transitive and symmetric relationship. That is, Clara is a sister of Carol, Carol is a sister of Cynthia, so Clara is a sister of Cynthia.



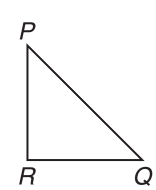
34. What other relationships in a family have reflexive, symmetric, or transitive relationships? Explain why. Remember that the child or children of each person are listed beneath that person's name. Consider relationships such as first cousin, ancestor or descendent, aunt or uncle, sibling, or any other relationship.

35. Construct your family tree on one or both sides of your family and identify the reflexive, symmetric, or transitive relationships.

36. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson.
How is mathematical evidence similar to evidence in law?
Include the following in your answer:
- a description of how evidence is used to influence jurors' conclusions in court, and
- a description of the evidence used to make conclusions in mathematics.

Standardized Test Practice

37. In $\triangle PQR$, $m\angle P = m\angle Q$ and $m\angle R = 2(m\angle Q)$. Find $m\angle P$ if $m\angle P + m\angle Q + m\angle R = 180$.



(A) 30 (B) 45
(C) 60 (D) 90

38. **ALGEBRA** If 4 more than x is 5 less than y, what is x in terms of y?

(A) $y - 1$ (B) $y - 9$ (C) $y + 9$ (D) $y - 5$

Maintain Your Skills

Mixed Review

39. **CONSTRUCTION** There are four buildings on the Medfield High School Campus, no three of which stand in a straight line. How many sidewalks need to be built so that each building is directly connected to every other building? *(Lesson 2-5)*

Determine whether the stated conclusion is valid based on the given information. If not, write *invalid*. Explain your reasoning. *A number is divisible by 3 if it is divisible by 6.* *(Lesson 2-4)*

40. **Given:** 24 is divisible by 6. **Conclusion:** 24 is divisible by 3.
41. **Given:** 27 is divisible by 3. **Conclusion:** 27 is divisible by 6.
42. **Given:** 85 is not divisible by 3. **Conclusion:** 85 is not divisible by 6.

Write each statement in if-then form. *(Lesson 2-3)*

43. "Happy people rarely correct their faults." (*La Rochefoucauld*)
44. "If you don't know where you are going, you will probably end up somewhere else." (*Laurence Peters*)
45. "A champion is afraid of losing." (*Billie Jean King*)
46. "If we would have new knowledge, we must get a whole new world of questions." (*Susanne K. Langer*)

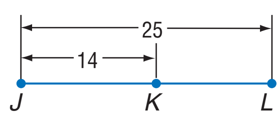
Find the precision for each measurement. *(Lesson 1-2)*

47. 13 feet
48. 5.9 meters
49. 74 inches
50. 3.1 kilometers

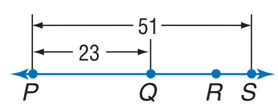
Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find the measure of each segment.**
(To review ***segment measures****, see Lesson 1-2.)*

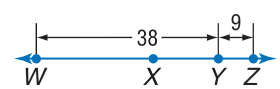
51. $\overline{KL}$



52. $\overline{QS}$



53. $\overline{WZ}$

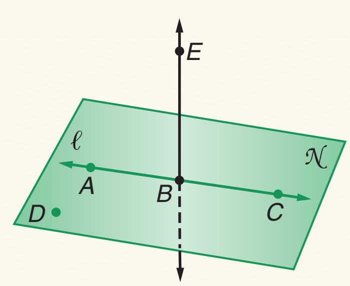


Practice Quiz 2

Lessons 2-4 through 2-6

1. Determine whether statement (3) follows from statements (1) and (2) by the Law of Detachment or the Law of Syllogism. If it does, state which law was used. If it does not, write *invalid*. *(Lesson 2-4)*
   (1) If n is an integer, then n is a real number.
   (2) n is a real number.
   (3) n is an integer.

In the figure at the right, *A*, *B*, and *C* are collinear. Points *A*, *B*, *C*, and *D* lie in plane *N*. State the postulate or theorem that can be used to show each statement is true. *(Lesson 2-5)*



2. *A*, *B*, and *D* determine plane *N*.
3. $\overleftrightarrow{BE}$ intersects $\overleftrightarrow{AC}$ at *B*.
4. ℓ lies in plane *N*.
5. **PROOF** If $2(n - 3) + 5 = 3(n - 1)$, prove that $n = 2$. *(Lesson 2-6)*