Adding and Subtracting Polynomials

Adding polynomials is just a matter of combining like terms, with some order of operations considerations thrown in. As long as you're careful with the minus signs, and don't confuse addition and multiplication, you should do fine.

There are a couple formats for adding and subtracting, and they hearken back to earlier times, when you were adding and subtracting just plain old numbers. First, you learned addition "horizontally", like this: $6 + 3 = 9$.
You can add polynomials in the same way, grouping like terms and then simplifying.

- **Simplify $(2x + 5y) + (3x - 2y)$**

  I'll clear the parentheses, group like terms, and then simplify:

  $$\begin{aligned} &(2x + 5y) + (3x - 2y) \\ &= 2x + 5y + 3x - 2y \\ &= 2x + 3x + 5y - 2y \\ &= \mathbf{5x + 3y} \end{aligned}$$

Horizontal addition works fine for simple examples. But when you were adding plain old numbers, you didn't generally try to add 432 and 246 horizontally; instead, you would "stack" them vertically, one on top of the other, and then add down the columns:

$$\begin{array}{r} 432 \\ \underline{+246} \\ 678 \end{array}$$

You can do the same thing with polynomials. This is how the above simplification exercise looks when it is done "vertically":

- **Simplify $(2x + 5y) + (3x - 2y)$**

  I'll put each variable in its own column; in this case, the first column will be the x-column, and the second column will be the y-column:

  $$\begin{array}{c} 2x + 5y \\ \underline{3x - 2y} \\ 5x + 3y \end{array}$$

  I get the same solution vertically as I got horizontally: $\mathbf{5x + 3y}$.

The format you use, horizontal or vertical, is a matter of taste (unless the instructions explicitly tell you otherwise). Given a choice, you should use whichever format that you're more comfortable and successful with. Note that, for simple additions, horizontal addition (so you don't have to rewrite the problem) is probably simplest, but, once the polynomials get complicated, vertical is probably safest bet (so you don't "drop", or lose, terms and minus signs).

- **Simplify $(3x^3 + 3x^2 - 4x + 5) + (x^3 - 2x^2 + x - 4)$**

  I can add horizontally:

  $$\begin{aligned} &(3x^3 + 3x^2 - 4x + 5) + (x^3 - 2x^2 + x - 4) \\ &= 3x^3 + 3x^2 - 4x + 5 + x^3 - 2x^2 + x - 4 \\ &= 3x^3 + x^3 + 3x^2 - 2x^2 - 4x + x + 5 - 4 \\ &= \mathbf{4x^3 + 1x^2 - 3x + 1} \end{aligned}$$

  ...or vertically:

  $$\begin{array}{rrrr} 3x^3 & +\ 3x^2 & -\ 4x & +\ 5 \\ \underline{x^3} & \underline{-\ 2x^2} & \underline{+\ \ x} & \underline{-\ 4} \\ 4x^3 & +\ 1x^2 & -\ 3x & +\ 1 \end{array}$$

  Either way, I get the same answer: $\mathbf{4x^3 + 1x^2 - 3x + 1}$.

Note that each column in the vertical addition above contains only one degree of x: the first column was the x^3 column, the second column was the x^2 column, the third column was the x column, and the fourth column was the constants column. This is analogous to having a thousands column, a hundreds column, a tens column, and a ones column when doing strictly-numerical addition.

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- **Simplify $(7x^2 - x - 4) + (x^2 - 2x - 3) + (-2x^2 + 3x + 5)$**
  It's perfectly okay to have to add three or more polynomials at once. I'll just go slowly and do each step throroughly, and it should work out right.
  Adding horizontally:

$$
\begin{aligned}
&(7x^2 - x - 4) + (x^2 - 2x - 3) + (-2x^2 + 3x + 5)\\
&= 7x^2 - x - 4 + x^2 - 2x - 3 + -2x^2 + 3x + 5\\
&= 7x^2 + \mathbf{1}x^2 - 2x^2 - \mathbf{1}x - 2x + 3x - 4 - 3 + 5\\
&= 8x^2 - 2x^2 - 3x + 3x - 7 + 5\\
&= \mathbf{6x^2 - 2}
\end{aligned}
$$

Note the 1's in the third line. Any time you have a variable without a coefficient, there is an "understood" 1 as the coefficient. If you find it helpful to write that 1 in, then do so.

Adding vertically:

```
 7x² -   x - 4
   x² - 2x - 3
-2x² + 3x + 5
-------------
 6x²       - 2
```

Either way, I get the same answer: $\mathbf{6x^2 - 2}$

- **Simplify $(x^3 + 5x^2 - 2x) + (x^3 + 3x - 6) + (-2x^2 + x - 2)$**
  Horizontally:

$$
\begin{aligned}
&(x^3 + 5x^2 - 2x) + (x^3 + 3x - 6) + (-2x^2 + x - 2)\\
&= x^3 + 5x^2 - 2x + x^3 + 3x - 6 + -2x^2 + x - 2\\
&= x^3 + x^3 + 5x^2 - 2x^2 - 2x + 3x + x - 6 - 2\\
&= \mathbf{2x^3 + 3x^2 + 2x - 8}
\end{aligned}
$$

When you add large numbers, there are sometimes zeroes in the numbers, such as:

```
  1002
+  560
------
  1562
```

The zeroes in "1002" stand for "zero hundreds" and "zero tens". They are what is called "placeholders", indicating that there are no hundreds or tens. If you didn't include those zeroes in the numerical expression, you'd have just in the top line "12", which isn't what you mean. The zeroes keep things lined up properly. When you vertically add polynomials that skip some of the degrees of x, you need to leave gaps, so the terms line up properly.

Vertically:

```
 x³ + 5x² - 2x
 x³       + 3x - 6
    - 2x² +  x - 2
------------------
2x³ + 3x² + 2x - 8
```

Either way, I get the same answer: $\mathbf{2x^3 + 3x^2 + 2x - 8}$

Subtracting polynomials is quite similar to adding polynomials, but you have that pesky minus sign to deal with. Here are some examples, done both horizontally and vertically:

- **Simplify $(x^3 + 3x^2 + 5x - 4) - (3x^3 - 8x^2 - 5x + 6)$**
  The first thing I have to do is take that negative through the parentheses. Some students find it helpful to put a "1" in front of the parentheses, to help them keep track of the minus sign:
  Horizontally:

$$
\begin{aligned}
&(x^3 + 3x^2 + 5x - 4) - (3x^3 - 8x^2 - 5x + 6)\\
&= (x^3 + 3x^2 + 5x - 4) - \mathbf{1}(3x^3 - 8x^2 - 5x + 6)\\
&= (x^3 + 3x^2 + 5x - 4) - \mathbf{1}(3x^3) - \mathbf{1}\,(-8x^2) - \mathbf{1}(-5x) - \mathbf{1}(6)
\end{aligned}
$$

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$= x^3 + 3x^2 + 5x - 4 - 3x^3 + 8x^2 + 5x - 6$

$= x^3 - 3x^3 + 3x^2 + 8x^2 + 5x + 5x - 4 - 6$

$= \mathbf{-2x^3 + 11x^2 + 10x - 10}$

Vertically:

$$\begin{array}{r} x^3 + 3x^2 + 5x - 4 \\ \underline{-\ (3x^3 - 8x^2 - 5x + 6)} \end{array}$$

In the horizontal case, you may have noticed that running the negative through the parentheses changed the sign on each term inside the parentheses. The shortcut here is to not bother writing in the subtaction sign or the parentheses; instead, you just change all the signs in the second row.

I'll change all the signs in the second row (shown in red below), and add down:

$$\begin{array}{r} x^3 + 3x^2 + 5x - 4 \\ \underline{-3x^3 + 8x^2 + 5x - 6} \\ -2x^3 + 11x^2 + 10x - 10 \end{array}$$

Either way, I get the answer: $\mathbf{-2x^3 + 11x^2 + 10x - 10}$

- **Simplify $(6x^3 - 2x^2 + 8x) - (4x^3 - 11x + 10)$**

Horizontally:

$(6x^3 - 2x^2 + 8x) - (4x^3 - 11x + 10)$

$= (6x^3 - 2x^2 + 8x) - \mathbf{1}(4x^3 - 11x + 10)$

$= (6x^3 - 2x^2 + 8x) - \mathbf{1}(4x^3) - \mathbf{1}(-11x) - \mathbf{1}(10)$

$= 6x^3 - 2x^2 + 8x - 4x^3 + 11x - 10$

$= 6x^3 - 4x^3 - 2x^2 + 8x + 11x - 10$

$= \mathbf{2x^3 - 2x^2 + 19x - 10}$

Vertically: I'll write out the polynomials, leaving gaps as necessary:

$$\begin{array}{rrrr} 6x^3 & - 2x^2 & + 8x & \\ \underline{4x^3} & & \underline{- 11x} & \underline{+ 10} \end{array}$$

Then I'll change the signs in the second line, and add:

$$\begin{array}{rrrr} 6x^3 & - 2x^2 & + 8x & \\ \underline{-4x^3} & & \underline{+ 11x} & \underline{- 10} \\ 2x^3 & - 2x^2 & + 19x & - 10 \end{array}$$

Either way, I get the answer: $\mathbf{2x^3 - 2x^2 + 19x - 10}$

Adding and Subtracting Polynomials

Add or Subtract each of the following. Show your work in a vertical format as on the notes.

1. $(4x^3 - 3x^2 + 2x - 5) + (2x^3 - x^2 + 3x - 2)$

2. $(3x^2 + 4x - 1) + (x^3 - 5x^2 - 2)$

3. $(10x^3 - x^2 + 2x - 1) + (4x^2 + 5x - 1)$

4. $(x^3 - x^2 + x - 1) + (2x^3 - 3x^2 + 5x - 7)$

5. $(42x^3 - 31x^2 + 23x - 17) + (29x^3 - 31x^2 + 13x - 12)$

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6. $(4x^3 - 3x^2 + 2x - 5) - (2x^3 - x^2 + 3x - 2)$

7. $(3x^2 + 4x - 1) - (x^3 - 5x^2 - 2)$

8. $(10x^3 - x^2 + 2x - 1) - (4x^2 + 5x - 1)$

9. $(x^3 - x^2 + x - 1) - (2x^3 - 3x^2 + 5x - 7)$

10. $(42x^3 - 31x^2 + 23x - 17) - (29x^3 - 31x^2 + 13x - 12)$