REVIEW EXAMPLES

1. Quadrilateral *ABCD* has vertices *A*(–1, 3), *B*(3, 5), *C*(4, 3), and *D*(0, 1). Is *ABCD* a rectangle? Explain how you know.

Solution:

First determine whether or not the figure is a parallelogram. If the figure is a parallelogram, then the diagonals bisect each other. If the diagonals bisect each other, then the midpoints of the diagonals are the same point. Use the midpoint formula to determine the midpoints for each diagonal.

$$\text{Midpoint } \overline{AC}: \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-1+4}{2}, \frac{3+3}{2}\right) = \left(\frac{3}{2}, \frac{6}{2}\right) = (1.5, 3)$$

$$\text{Midpoint } \overline{BD}: \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{3+0}{2}, \frac{5+1}{2}\right) = \left(\frac{3}{2}, \frac{6}{2}\right) = (1.5, 3)$$

The diagonals have the same midpoint; therefore, the diagonals bisect each other and the figure is a parallelogram.

A parallelogram with congruent diagonals is a rectangle. Determine whether or not the diagonals are congruent.

Use the distance formula to find the length of the diagonals:

$$AC = \sqrt{(4-(-1))^2 + (3-3)^2} = \sqrt{(5)^2 + (0)^2} = \sqrt{25+0} = \sqrt{25} = 5$$

$$BD = \sqrt{(0-3)^2 + (1-5)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

The diagonals are congruent because they have the same length.

The figure is a parallelogram with congruent diagonals, so the figure is a rectangle.

2. Circle *C* has a center of (–2, 3) and a radius of 4. Does point (–4, 6) lie on circle *C*?

Solution:

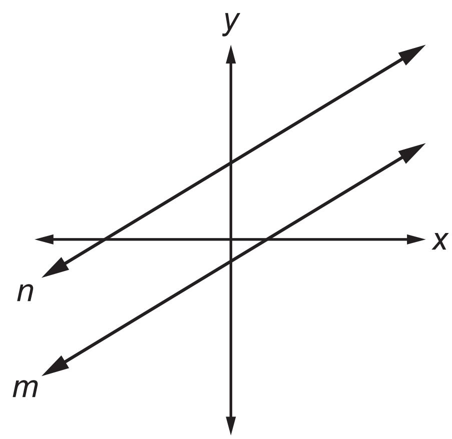
The distance from any point on the circle to the center of the circle is equal to the radius. Use the distance formula to find the distance from (–4, 6) to the center (–2, 3). Then see if it is equal to the radius, 4.

$\sqrt{(-4-(-2))^2 + (6-3)^2}$	Substitute the coordinates of the points in the distance formula.
$\sqrt{(-2)^2 + (3)^2}$	Evaluate within parentheses.
$\sqrt{4+9}$	Evaluate the exponents.
$\sqrt{13}$	Add.

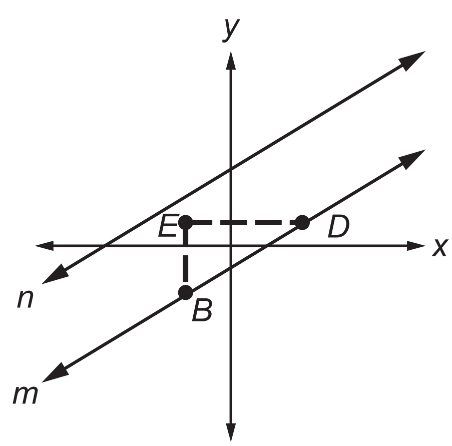
The distance from (–4, 6) to (–2, 3) is not equal to the radius, so (–4, 6) does not lie on the circle. (In fact, since $\sqrt{13} < 4$, the distance is less than the radius, so the point lies inside of the circle.)

3. Follow the steps below to prove that if two nonvertical lines are parallel, then they have equal slopes.

   a. Let the straight lines *n* and *m* be parallel. Sketch these on a coordinate grid.

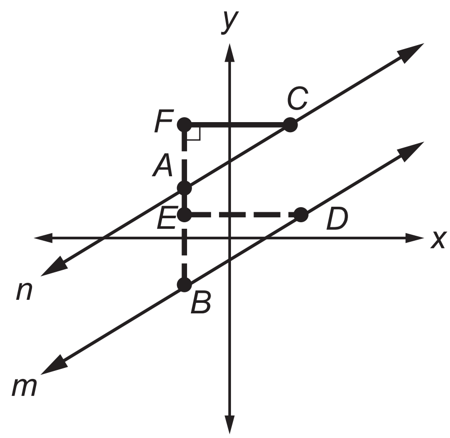


   b. Plot any points *B* and *D* on line *m* and the point *E* so that segment *BE* is the rise and segment *ED* is the run of the slope of line *m*. (A straight line can have only one slope.)



That is, slope of line *m* is $\frac{BE}{ED}$.

c. Draw the straight line *BE* so that it intersects line *n* at point *A* and extends to include point *F* such that segment *FC* is perpendicular to *BE*.



d. What is the slope of line *n*?

Solution:

The slope is $\frac{AF}{FC}$.

e. Line *BF* is the ____________________ of lines *m* and *n*, so ∠*EBD* and ∠*FAC* are ______________________________________ angles, so ∠*EBD* ____ ∠*FAC*.

Solution:

transversal, corresponding, congruent

f. Why is it true that $\angle DEB \cong \angle AFC$?

Solution:

The angles are right angles.

g. Now, △*DEB* and △*CFA* are similar, so the ratio of their sides is proportional. Write the proportion that relates the vertical leg to the horizontal leg of the triangles.

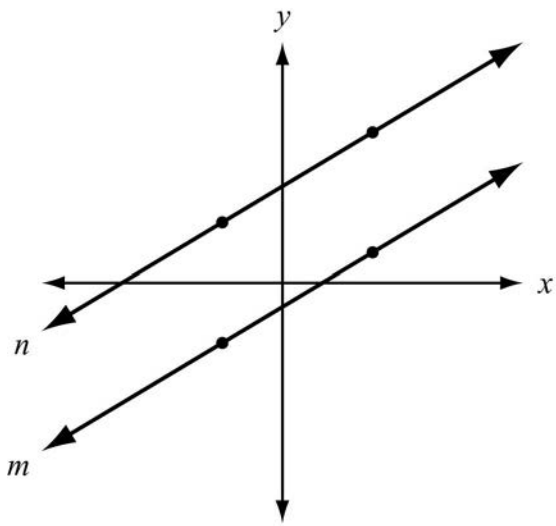
Solution:

$$\frac{BE}{ED} \cong \frac{AF}{FC}$$

h. Note that this proportion shows the slope of line *m* is the same as the slope of line *n*. Therefore, parallel lines have the same slope.

4. Prove that if two nonvertical lines have equal slopes, then they are parallel.

Solution:



Use a proof by contradiction. Assume that the lines have equal slopes but are not parallel—that is, assume the lines intersect. If you can show this is not true, it is equivalent to proving the original statement.

Write the equations for both lines. The slopes are the same, so use m for the slope of each line. The two lines are different, so $b_1 \neq b_2$.

Equation for line n: $y = mx + b_1$

Equation for line m: $y = mx + b_2$

Solve a system of equations to find the point of intersection. Both equations are solved for y, so use substitution.

$$mx + b_1 = mx + b_2$$

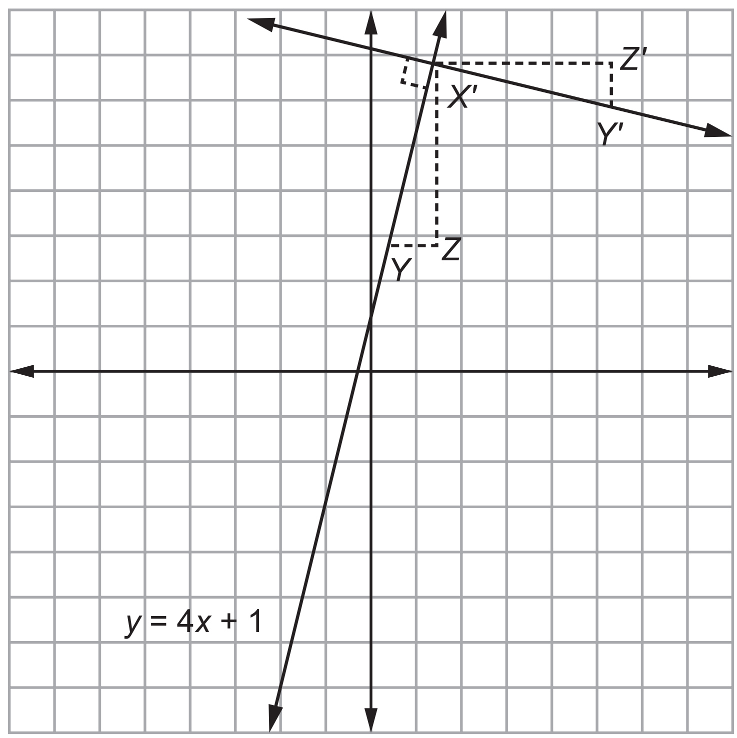
$$b_1 = b_2$$

Because it was assumed that $b_1 \neq b_2$, this is a contradiction. So the original statement is true—if two nonvertical lines have equal slopes, then they are parallel.

5. The line p is represented by the equation $y = 4x + 1$. What is the equation of the line that is perpendicular to line p and passes through the point (8, 5)?

Solution:

On a coordinate grid, graph the line $y = 4x + 1$. Locate points Y and Z on the line $y = 4x + 1$ such that the slope of $\frac{XZ}{YZ} = 4$. Rotate triangle XYZ around point X 90°. Name the new triangle $X'Y'Z'$. X' and Y' lie on line $X'Y'$ so that line $X'Y'$ passes through the point (8, 5) and has a slope of $-\frac{1}{4}$.



The slope-intercept form of the equation of a line is $y = mx + b$. Substitute $-\frac{1}{4}$ for m. The line perpendicular to line p passes through (8, 5), so substitute 8 for x and 5 for y. Solve for b.

$$5 = -\frac{1}{4}(8) + b$$

$$5 = -2 + b$$

$$7 = b$$

The equation of the line perpendicular to line p and that passes through (8, 5) is $y = -\frac{1}{4}x + 7$.

6. For what value of n are the lines $7x + 3y = 8$ and $nx + 3y = 8$ perpendicular?

Solution:

The two lines will be perpendicular when the slopes are opposite reciprocals.

First, find the slope of the line $7x + 3y = 8$.

$$7x + 3y = 8$$
$$3y = -7x + 8$$
$$y = -\frac{7}{3}x + \frac{8}{3}$$

The slope is $-\frac{7}{3}$.

Next, find the slope of the line $nx + 3y = 8$, in terms of n.

$$nx + 3y = 8$$
$$3y = -nx + 8$$
$$y = -\frac{n}{3}x + \frac{8}{3}$$

The slope is $-\frac{n}{3}$.

The opposite reciprocal of $-\frac{7}{3}$ is $\frac{3}{7}$. Find the value of n that makes the slope of the second line $\frac{3}{7}$.

$$-\frac{n}{3} = \frac{3}{7}$$
$$-n = \frac{9}{7}$$
$$n = -\frac{9}{7}$$

When $n = -\frac{9}{7}$, the two lines are perpendicular.

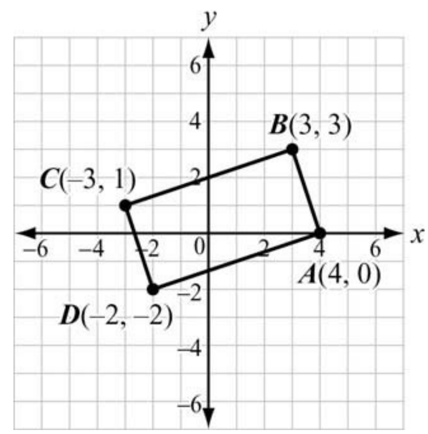
7. Quadrilateral *ABCD* has vertices *A*(4, 0), *B*(3, 3), *C*(–3, 1), and *D*(–2, –2). Prove that *ABCD* is a rectangle.

Solution:

The slopes of the sides are:

$$\overline{AB}: \frac{3-0}{3-4} = \frac{3}{-1} = -3 \qquad \overline{BC}: \frac{1-3}{-3-3} = \frac{-2}{-6} = \frac{1}{3}$$

$$\overline{CD}: \frac{-2-1}{-2+3} = \frac{-3}{1} = -3 \qquad \overline{DA}: \frac{0+2}{4+2} = \frac{2}{6} = \frac{1}{3}$$



$\overline{AB} \parallel \overline{CD}$ because they have equal slopes.

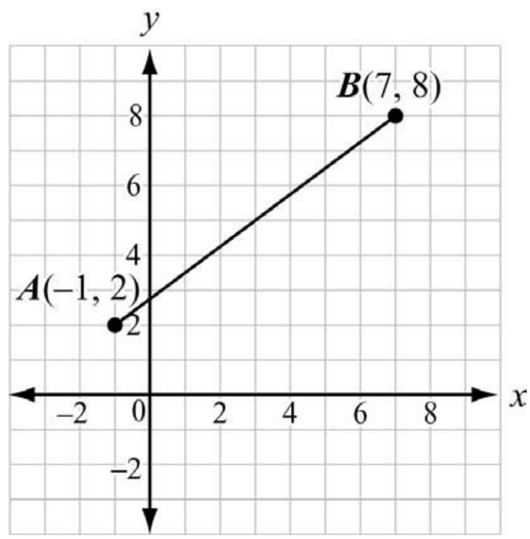
$\overline{BC} \parallel \overline{DA}$ because they have equal slopes.

So *ABCD* is a parallelogram because both pairs of opposite sides are parallel.

$\overline{AB} \perp \overline{BC}$ because the product of their slopes is –1: $-3 \cdot \frac{1}{3} = -1$.

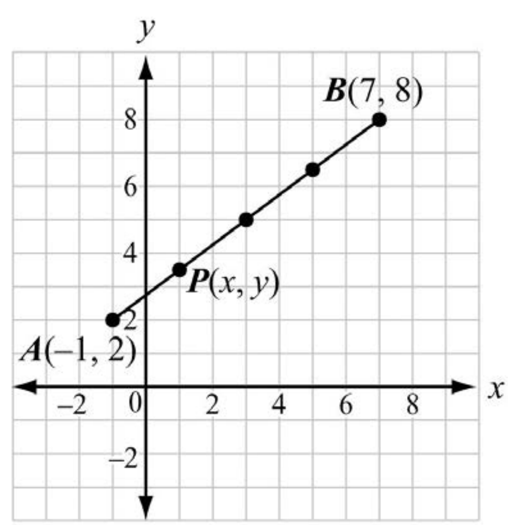
Therefore, *ABCD* is a rectangle because it is a parallelogram with a right angle.

8. Given the points $A(-1, 2)$ and $B(7, 8)$, find the coordinates of the point P on directed line segment $\overline{AB}$ that partitions $\overline{AB}$ in the ratio 1:3.



Solution:

Point P partitions $\overline{AB}$ in the ratio 1:3 if P is on $\overline{AB}$. This means that you need to divide $\overline{AB}$ into 4 equal parts, since $\overline{AP}$ = one-fourth of $\overline{AB}$.



Let $P(x, y)$ be on $\overline{AB}$. Solve two equations to find x and y, where (x_1, y_1) is the starting point, (x_2, y_2) is the ending point, and $\frac{a}{(a+b)} = \frac{1}{4}$.

$$(x, y) = \left(x_1 + \frac{a}{a+b}(x_2 - x_1),\ \ y_1 + \frac{a}{a+b}(y_2 - y_1)\right)$$

$$(x, y) = \left(-1 + \frac{1}{4}(7 - (-1)),\ \ 2 + \frac{1}{4}(8 - 2)\right)$$

$$x = -1 + \frac{1}{4}(7-(-1)) \qquad y = 2 + \frac{1}{4}(8-2)$$

$$x = -1 + \frac{1}{4}(8) \qquad y = 2 + \frac{1}{4}(6)$$

$$x = -1 + 2 \qquad y = 2 + \frac{3}{2}$$

$$x = 1 \qquad y = \frac{7}{2}$$

The coordinates of P are $\left(1, \frac{7}{2}\right)$.

Here is another method for partitioning segment AB in the ratio 1:3 from $A(x_1, y_1)$ to $B(x_2, y_2)$.

$$(x, y) = \frac{bx_1 + ax_2}{b+a}, \frac{by_1 + ay_2}{b+a}$$

$$= \frac{3(-1) + 1(7)}{3+1}, \frac{3(2) + 1(8)}{3+1}$$

$$= \frac{-3+7}{4}, \frac{6+8}{4}$$

$$= \frac{4}{4}, \frac{14}{4}$$

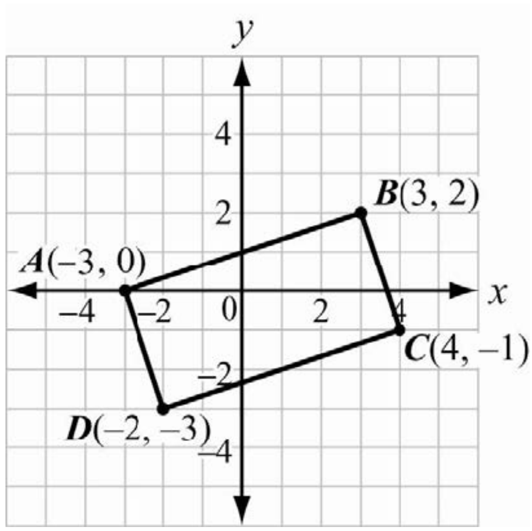
$$= \left(1, \frac{7}{2}\right)$$

Important Tip

✍ Be careful when using directed line segments. If point P partitions $\overline{AB}$ in the ratio 1:3, then point P partitions $\overline{BA}$ in the ratio 3:1.

9. Find the area of rectangle *ABCD* with vertices *A*(–3, 0), *B*(3, 2), *C*(4, –1), and *D*(–2, –3).

Solution:



One strategy is to use the distance formula, $d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$, to find the length and width of the rectangle.

$$AB = \sqrt{\left(3-(-3)\right)^2 + \left(2-0\right)^2} = \sqrt{36+4} = \sqrt{40}$$

$$BC = \sqrt{\left(4-3\right)^2 + \left(-1-2\right)^2} = \sqrt{1+9} = \sqrt{10}$$

The length of the rectangle is usually considered to be the longer side. Therefore, the length of the rectangle is $\sqrt{40}$ and the width is $\sqrt{10}$.

NOTE: Other strategies are possible to find the length of AB and BC such as using the Pythagorean theorem.

Use the area formula.

$$A = lw$$
$$A = \left(\sqrt{40}\right)\left(\sqrt{10}\right)$$
$$A = \sqrt{400}$$
$$A = 20$$

The area of the rectangle is 20 square units.